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XI.—Quantum Theory of the Chemical Bond.

By **C. A. Coulson**, University College, Dundee. *An Address delivered before the Society on February 3, 1941.* (With Eighteen Text-figures.)

(MS. received June 11, 1941.)

1. Introduction.

THE wave mechanics introduced in 1926 by Schrödinger has proved very fruitful in explaining many features of atomic structure, and in practically every important respect our knowledge of the motions and energies of the electrons which move outside the nucleus of an atom is satisfactory. The same is not true of molecular structure: the mathematical complications involved in a precise detailed description of the orbits are so great that they have been overcome only in the case of one molecule—molecular hydrogen. So in this review of some of the investigations of the last few years we shall usually be speaking in terms of approximations; indeed right at the start we meet two different main avenues of approximation, known as the molecular orbital and electron-pair methods respectively. We confine ourselves here to the former of these, the molecular orbital approximation, not because it is the better (neither is satisfactory, and the existence of the two complementary approximations is an indication of our partial failure to solve the problem), but because it is easier to maintain one point of view consistently throughout this account. The molecular orbital method is associated particularly with the names of Hund, Hückel, Mulliken, and Lennard-Jones, and the electron-pair method with the names of Heitler, London, Pauling, Slater, and Van Vleck.

2. Atomic Orbits.

Since molecules are formed out of atoms and, as we shall see, molecular orbits are themselves compounded out of atomic orbits, it will be convenient to review briefly the wave mechanical description of electrons in an atom. According to the theory initiated by Hartree, the motions of these electrons are governed by the following three principles:—

(*a*) Each electron is assigned to a definite orbit, which is described by a wave function $\psi(x, y, z)$, in which x, y, z are the co-ordinates of the electron relative to the nucleus, and ψ itself has to be one of a definite set of allowed functions, which, since they define the orbit, may be called

Atomic Orbitals. ψ is found as the solution of a certain differential equation, the Schrödinger Wave Equation. $\psi^2(x, y, z)$ or, if ψ is complex, $|\psi(x, y, z)|^2$ represents the probability that the electron will be found in a certain region of space around (x, y, z). If ψ is large at any point irrespective of its sign (which has no physical meaning), the electron is likely to be found in that region.

(*b*) Each of these wave functions ψ has its own appropriate energy

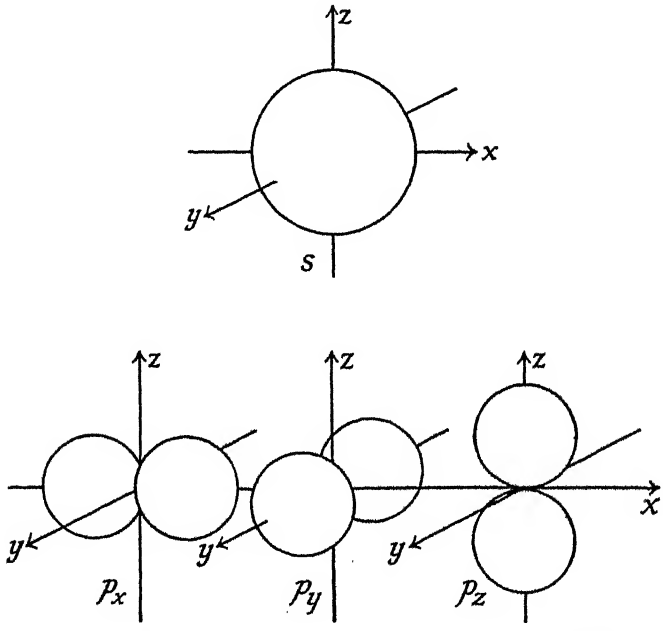


FIG. 1.—Atomic *s* and *p* orbits as found in Hydrogen and Carbon.

value and, if this is suitably determined, the total energy is merely the sum of the energies of the constituent orbitals.

(*c*) In addition to its wave function $\psi(x, y, z)$, an electron has a spin, which must have one of two values ($\pm\frac{1}{2}$ in suitable units), and the Pauli Exclusion Principle tells us that not more than two electrons in any atom (or molecule) may have the same orbit; further, if there are two, they must have their spins different, or opposed to each other. Two such electrons having the same ψ but opposed spins form a very stable combination (we speak of them as paired together); they are unreactive and usually exert a repulsion upon any other electrons near them.

We shall be mainly concerned in this account with Hydrogen and Carbon. Fig. 1 (after Pauling) shows a rough drawing of the electron patterns in these atoms. In Hydrogen there is just one electron around the central nucleus and this electron has a spherically symmetrical

pattern, similar to that marked *s*. It is known as the 1*s* orbital, and is characterised by the fact that the electron is most likely to be found around or inside the sphere shown (whose radius is approximately the atomic radius of the atom). In Carbon, with six electrons, there are two inner or K-shell orbits which are similar to the Hydrogen 1*s* orbit, but much more contracted into the nucleus; these take no part in chemical binding and we need not concern ourselves with them. Next there are four L-shell electrons, whose orbits must be chosen from 2*s*, $2p_x$, $2p_y$, and $2p_z$. The 2*s* orbit is spherically symmetrical, and only slightly smaller than the Hydrogen 1*s* orbit; in the other three, shown as p_x, p_y, and p_z in the figure, which are also about the same size as the 2*s* orbit, the electron is practically confined within a region which resembles a dumb-bell; on account of this fact we may conveniently refer to these as "dumb-bell" orbits. They have a marked directional property; in fact ψ is positive in one of the preferred regions and negative in the other, having what is called a nodal plane, or zero value, between the two. The analytical form of the $2p_x$ wave function, for example, is $\psi(2p_x)=xe^{-cr}$, where c is a known constant and r is the distance from the nucleus; this shows that $x=0$ is a plane of zero value.

The precise way in which these four orbits are filled depends upon the nature of the bonds that Carbon is forming in any particular molecule. For since all three "dumb-bell" orbits have the same energy, we may take any linear combinations of them instead of $2p_x$, $2p_y$, and $2p_z$. Further, if the energy of the 2*s* orbital does not differ greatly from that of the 2*p* orbitals (the energy of binding of the 2*s* orbit is greater than that of the 2*p* orbits), we may include the 2*s* wave function in the new compound, or hybrid, orbitals. If the energies are considerably different, as in Oxygen, this hybridisation does not take place to any large extent. But in Carbon it happens quite easily, and there are at least two linear combinations of the 2*s* and 2*p* wave functions that are of the utmost importance. They are:

Type A (tetrahedral orbits).	Type B (trigonal orbits).
$\psi(t_1)=\psi(2s)+\psi(2p_x)+\psi(2p_y)+\psi(2p_z)$	$\psi(I) = \psi(2s)+\sqrt{2}\psi(2p_x)$
$\psi(t_2)=\psi(2s)+\psi(2p_x)-\psi(2p_y)-\psi(2p_z)$	$\psi(II) = \sqrt{2}\psi(2s)-\psi(2p_x)+\sqrt{3}\psi(2p_y)$
$\psi(t_3)=\psi(2s)-\psi(2p_x)+\psi(2p_y)-\psi(2p_z)$	$\psi(III) = \sqrt{2}\psi(2s)-\psi(2p_x)-\sqrt{3}\psi(2p_y)$
$\psi(t_4)=\psi(2s)-\psi(2p_x)-\psi(2p_y)+\psi(2p_z)$	$\psi(IV) = \psi(2p_z)$

Type A we may call the tetrahedral type, because all four wave patterns are identical, with a form resembling that shown in fig. 2; in one of these orbits the electron is very concentrated in a particular direction, and the wave pattern is symmetrical about this direction; there is little probability of finding the electron anywhere else than in

this region. A very important property of these orbits is that they point towards the four vertices of a regular tetrahedron surrounding the Carbon atom. We are not surprised, therefore, to discover that the characteristic tetravalency of Carbon is associated with the fact that, when it is ready for the formation of a saturated molecule such as CH_4 or C_2H_6, we find one electron in each of the four orbits $t_1 \ldots t_4$.

The second type, which we may call the trigonal type, resembles A in that the first three of the hybrid orbitals are very similar to the tetrahedral functions, having a marked directional property; but these three

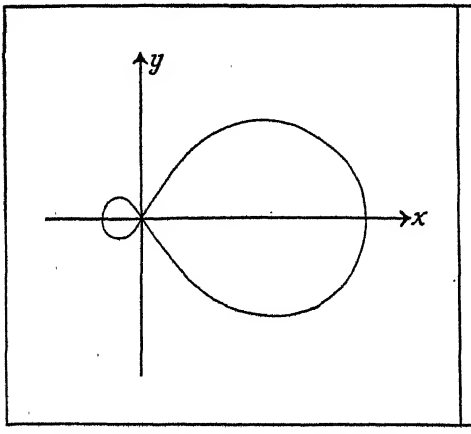


FIG. 2.—A single tetrahedral orbital.

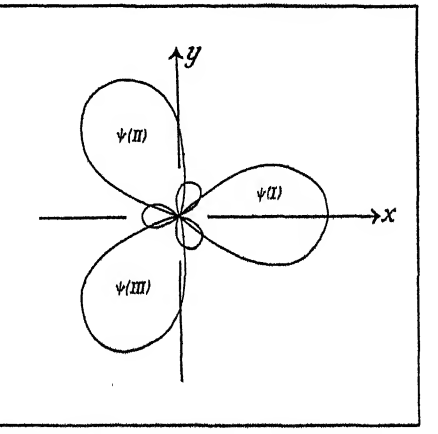


FIG. 3.—The three co-planar trigonal orbitals.

principal directions are all in a plane (the xy plane), and the angles between them are 120°. These are shown in fig. 3. The remaining orbit of this series is an undisturbed "dumb-bell" orbit, with its axis perpendicular to the plane of the other three. Thus in this configuration there are three equivalent directions in the plane, and we are not surprised to learn that when the Carbon atom is about to form an unsaturated molecule, such as Ethylene C_2H_4, in which the angles between the bonds are 120°, there is one electron in each of the trigonal orbits $I \ldots IV$.

3. THE NORMAL SINGLE BOND.

When we pass from studying the electrons in atoms to discuss the formation of molecules, we have to consider how the electrons will move in the presence, not of one, but of two or more nuclei. The three principles (*a*), (*b*), and (*c*) which we outlined in § 2 for the atomic orbits still hold for molecular orbits. That is:

(*a*) Each electron is assigned a wave function ψ which defines the orbit. These wave functions, or *molecular orbitals*, as they are called,

to distinguish them from the *atomic orbitals*, are no longer monocentric, but are polycentric, since an electron which takes part in molecule formation is not confined to one nucleus. $\psi^2(x, y, z)$ still represents the probability of finding the particular electron around the point (x, y, z).

(*b*) Each ψ is associated with a definite energy value.

(*c*) Each electron has a spin, and obeys the Pauli Exclusion Principle, just as in the case of an atom.

To these three principles we may add three more:

(*d*) In the neighbourhood of any nucleus a molecular orbit resembles one of the allowed atomic orbits. This follows from the fact that when the electron is near a particular nucleus, the chief forces on it arise from this nucleus and the other electrons round this nucleus, so that the solution of the wave equation in this neighbourhood will resemble an atomic orbital. In this sense, then, molecular orbitals may be regarded as built up out of atomic orbitals.

(*e*) The energy of a molecular orbit is lowest (*i.e.* binding energy is greatest) when the atomic orbits which it resembles overlap as much as possible. This is the *Principle of Maximum Overlapping* and, as we shall see, immediately leads us to the fundamental results of stereochemistry.

(*f*) The five principles (*a*)–(*e*) hold for all molecular orbits. However, when we are dealing with molecules in which only single bonds occur, it is possible to show, though we shall not reproduce the analysis here, that a good enough approximation to the molecular orbits is obtained by supposing that they are bicentric; that is, they embrace two neighbouring nuclei only. Strictly, of course, we should allow them the possibility of migrating from one nucleus to any other, but it turns out that an electron which is helping to form a single bond between two atoms A and B has very little chance of moving away from the region between A and B on to other nuclei C, D . . . These electrons may be called *localised electrons*.

We may illustrate these principles by considering first the simplest of all molecules H_2. Here there are two electrons, and each is assigned the same wave function ψ, with opposed spins. Near one of the nuclei, A, ψ resembles the ordinary atomic orbital ψ_a, and near the other nucleus B it resembles ψ_b. We may take for its approximate form

$$\psi = \psi_a + \psi_b. \quad . \quad . \quad . \quad . \quad . \quad (1)$$

This wave function is represented in fig. 4, where, on the left, we have the two separate atomic orbits out of which the final orbit on the right is formed. Such a wave pattern is symmetrical about the line AB, and we

might refer to it pictorially as a "sausage-type" orbit, since the electrons are each confined to the same region whose shape resembles that of a sausage. These orbits are known as σ-type orbits. In fig. 5 there is drawn the probability function ψ^2 for points along the line of symmetry AB, which joins the nuclei. It is clear from this diagram that the

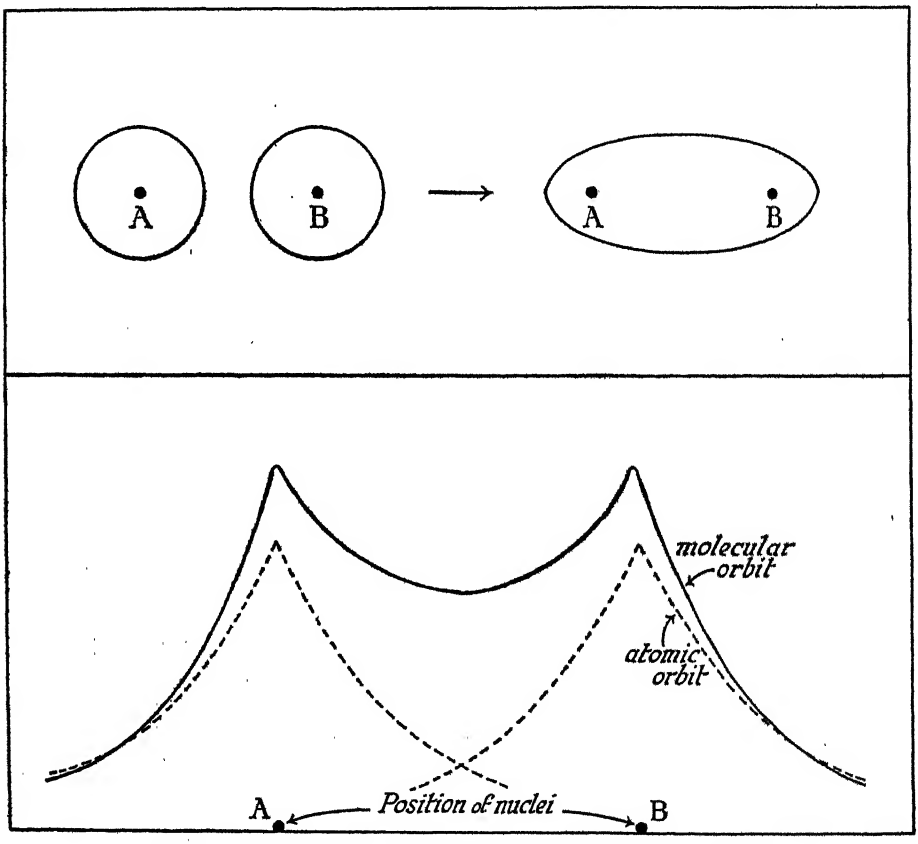


FIG. 4.—Formation of molecular orbit for H_2 out of atomic orbits.
FIG. 5.—Probability function ψ_2 along the axis of H_2.

electrons in this bond are closely localised in the region between A and B, with very little probability of moving away from the "bond direction." These two electrons, with identical orbits and opposed spins, form what we may call an *electron-pair*; they cause the molecule to repel other electrons, and indeed a certain amount of energy is usually needed to overcome this before the molecule can be made to react with other molecules.

The energies of these orbits may be calculated, and it turns out that they are lower than those of isolated Hydrogen atoms. Thus there is a

lowering of the total energy when two Hydrogen atoms join to form a molecule H_2. This, with some small corrections, is the binding energy.

The Hydrogen molecule-ion H_2^+, in which there is only one electron, is found in the discharge tube. This molecule will resemble H_2 in that the single electron has a bicentric orbit similar to that in fig. 4; but since this electron is not paired with a second electron to form a closed group, the ion will be very reactive. Indeed this is found to be the case, for it rapidly combines with other molecules and cannot be isolated.

If we had considered HCl instead of H_2 the situation would have been very similar. For there are just two unpaired electrons available for forming bonds; one of them is the Hydrogen $1s$ electron, and the other is the Chlorine $3p_x$ electron. (The $3p_x$ orbit resembles the $2p_x$ orbit of fig. 1, and the x direction is along the line of the nuclei.) These two electrons together form a σ-type molecular orbit which may be described, as in (1), by the expression

$$\psi = \lambda\psi(1s) + \mu\psi(3p_x). \quad . \quad . \quad . \quad . \quad (2)$$

The constants λ and μ may be found mathematically, but they are no longer equal since the two constituent atoms are different. The orbit is like a "sausage," with symmetry about the line of nuclei, but the "sausage" is fatter at one end (Cl) than at the other, corresponding to a greater chance of finding the electron near the more electronegative atom Cl. In fact, the ratio $\lambda : \mu$ may be immediately related to the dipole moment of HCl.

The simplest polyatomic molecule to illustrate the rules (*a*)–(*f*) is Water H_2O. The Oxygen atom has two inner K electrons which remain close around the nucleus and do not concern us further; there are six others, and two of them fill the $2s$ atomic orbital, forming a closed subgroup around the nucleus. Two others fill the $2p_z$ orbital (fig. 1) pointing in the z direction, and, being paired together, cannot take part in molecule building. The remaining two are one each in the $2p_x$ and $2p_y$ orbits (fig. 6). As we stated earlier, there is practically no hybridisation with Oxygen, so that the $2s$ and $2p$ wave functions do not mix. If we are to get maximum overlapping (rule (*e*)), we must put the two Hydrogen atoms along the x and y axes, as shown on the left in fig. 6. The resulting molecular orbits are shown on the right; two electrons each, with opposed spins, are allotted to each of the σ-type bonds shown. Each orbit resembles a Hydrogen $1s$ orbit near the H nucleus, and an Oxygen $2p$ orbit near the O nucleus, and is symmetrical about the line of the bond. We can see on this ground that the HOH angle should be about 90°; actually

repulsions between the two H atoms and a trace of hybridisation opens out the angle to about 105°.

A similar discussion could be given for ammonia NH_3. The N atom has three unpaired electrons, one each in the $2p_x$, $2p_y$, and $2p_z$ orbits. It forms three equivalent bonds with the H atoms, such that the HNH angles should all be 90°, and each bond involves two electrons in a σ-type orbit having opposed spins. Here again the H repulsions open out the HNH angles to rather more than 90°, making the molecule a somewhat flat pyramid.

We can deal with Carbon compounds in the same way. If we suppose that the four unpaired electrons in Carbon are one each in the four tetrahedral orbits (A) of § 2, it can form four equivalent bonds, and the Criterion of Maximum Overlapping shows that these are directed to the four corners of a regular tetrahedron, making angles of 109° 28′ with each other. In this way the familiar facts of stereochemistry in organic

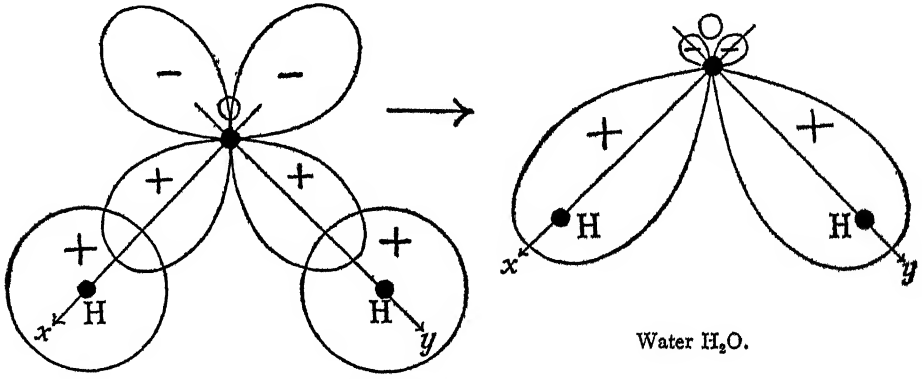


FIG. 6.—The Water molecule.

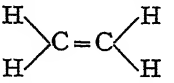
compounds involving only single bonds receive a natural explanation. If the four radicals which are bonded to the central Carbon atom are different among themselves, their mutual repulsions and different electron affinities will cause a slight dislocation of the tetrahedral angles, but it can be shown that this effect will not alter the angles by more than 2 or 3 degrees.

One more result concerning these single bonds follows from our previous discussion. In every case a single bond is represented by two electrons whose orbits are closely confined, or localised, in the region between two nuclei and, except in unusual circumstances, almost independent of the nature of the other atoms bonded to the two nuclei concerned. It follows from this that the energy of such a bond is almost independent of the other bonds present, and we are led to the idea of a

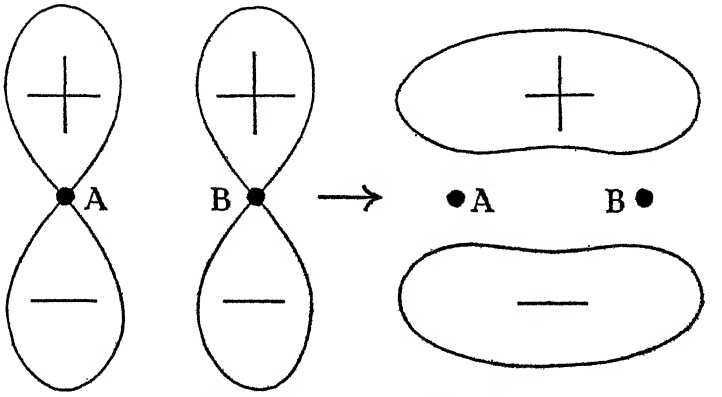
bond energy, characteristic of two bonded atoms, and practically constant for all molecules. In this way a table of bond energies could be drawn up, and it appears, naturally enough, that in these saturated molecules bond energies are constant and, apart from a few exceptional cases, are additive to give the total binding energy of the molecule.

4. The Double Bond.

A new situation arises when we come to the unsaturated molecules in which there occur double bonds; the characteristic case is Ethylene:



To explain the orbits of the electrons in this molecule we must go back to the trigonal wave functions already described in § 2. For if the two



Mobile electrons in Ethylene C_2H_4.

Fig. 7.—The Ethylene "double-streamers."

Carbon atoms are in the trigonal state we can soon form five "sausage" bonds, each of two electrons, such that there are three bonds from each Carbon atom: and the Criterion of Maximum Overlapping requires that these are all at angles of 120° with each other. This leaves us with one electron from each Carbon atom which is unpaired and not so far engaged in the formation of a bond. These two "loose" electrons are the dumb-bells perpendicular to the planes defined by the earlier orbitals, and the best that we can do is to pair these together somehow. Now if we arrange the two CH_2 planes so that they are coincident, the two dumb-bells are pointing parallel, and this is the direction in which they overlap most (fig. 7). It is true that the overlap is less than with the other bonds, and this accounts for the less stability of these orbits, but we cannot do better. If we now allow them to interact, these $2p_z$ electrons will form themselves

into molecular orbitals, as shown in the figure, in which the probability function resembles two streamers, one below and the other above, the plane of the C_2H_4 skeleton. It can easily be seen that if we divide this wave pattern into two halves by a plane perpendicularly bisecting AB, the two halves bear a close resemblance to the atomic $2p_z$ orbits from which the molecular orbit is formed. The wave function for this new kind of molecular orbital may be written down in a manner similar to that used for H_2. Thus near nucleus A the orbital resembles $\psi_a(2p_z)$, and near nucleus B it resembles $\psi_b(2p_z)$; its approximate description is therefore

$$\psi = \psi_a(2p_z) + \psi_b(2p_z). \quad . \quad . \quad . \quad . \quad (3)$$

It is precisely this streamer effect, in which the two electrons are paired with opposite spin, each having a wave function such as that in fig. 7, that converts a single bond into a double bond, and which is characteristic of every double bond. It is important to realise that both electrons have the same "double-streamer" wave function; it is not a case of the top streamer representing one electron and the bottom streamer the other electron. The two streamers go together; they are inseparable, and both together constitute that "extra" which, superposed on a single bond, converts it into a double bond. These streamers, which of course are very far from being symmetrical about the line joining the nuclei, are called a π-bond.

If this description of an Ethylene molecule is correct—that four of the bonds are of σ-type, each with two electrons, and the fifth has four electrons, two being in a σ-type orbit and the other two in a "double-streamer" π-type orbit—it is necessary that the molecule should be planar (otherwise the π-electrons cannot overlap as much as possible); also the angles between all the bonds should be 120°, and there should be some considerable resistance to twisting one of the CH_2 groups relative to the other. Every one of these inferences has received ample confirmation at the hands of the spectroscopists, crystallographers, and X-ray workers.

The double bond is thus very far from being two single bonds superposed, and, on account of the double-streamer electrons, it will have characteristic properties not possessed by single bonds. One of these is its high reactivity; this is explained by the fact that these streamer electrons are not so tightly bound as the other electrons because the atomic orbits from which they are built up do not overlap so much; consequently it is easier to disengage them from one another and link them to other atoms, to form a new molecule.

The triple bond, about which we shall not speak further here, differs

from the double bond simply in the addition of an extra pair of streamer electrons, but the plane of these new ones is perpendicular to that of the former pair. As a result we have a bond that is somewhat more symmetrical about the line joining the nuclei, and less resistant to twisting of the two halves relative to each other.

The description that has been given above for the C=C bond would apply just as well for other double bonds, as O=O or S=S, except that there are none of the hybrid single bonds present. With mixed bonds, however, such as C=O, the double streamers will be thrown more over on to one or other of the nuclei (with C=O, the O nucleus) on account of its greater electronegativity. In every case the electrons are allotted to orbits which are localised in the region of the two nuclei, and the normal, or pure, double bond, like the normal single bond, has a characteristic binding energy which is additive throughout the molecule.

5. Conjugated Compounds.

When we come to discuss more complicated compounds in which single and double bonds occur alternately, the dumb-bell electrons which

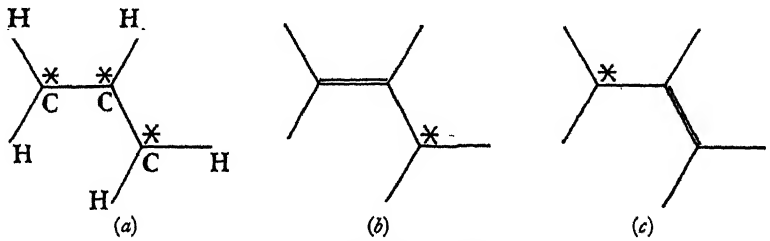


Fig. 8.—The Allyl radical.

we used in Ethylene to convert a single bond into a double bond become even more important. Let us consider the Allyl radical C_3H_5 (fig. 8). Each of the bonds marked in the diagram (*a*) will absorb two electrons, and will form a set of σ-type bonds. There remains one electron from each Carbon atom not so far engaged in molecule formation. If the angles of the bonds are 120° so that the Carbon atoms are in the familiar trigonal state, these free electrons are the dumb-bells perpendicular to the planes of the other bonds. We have supposed that all the atoms lie in a plane (a situation that has been confirmed in other similar molecules by spectroscopic analysis), so that these three dumb-bells point in parallel directions perpendicular to the plane of the paper; they are marked symbolically with a cross. The molecular orbits for these three electrons are different from any others yet described; for it turns out that these orbits have to embrace all three nuclei; they are tricentric orbits and we

cannot localise them in pairs to form a double bond between any two of the nuclei. In other words, these three electrons are free to move over the whole Carbon skeleton. For this reason they have been called mobile electrons. (Other names in current use are p_h-electrons, π-electrons, or unsaturation electrons.) In more complex molecules they may be able to move over as many as ten or a dozen nuclei, and they resemble very closely the conduction electrons of a metal. The binding effect of these mobile electrons is, indeed, intermediate between that of an ordinary molecule with its localised bonds, and a metal where there are no rigidly maintained bonds at all. It is this easy mobility of these electrons that accounts for the anomalous electric and magnetic susceptibilities of aromatic molecules, and also for the readiness with which influences may be transmitted from one part of the molecule to another. The ortho-para directing property of substituted benzenes finds a ready explanation in this manner.

Returning then to the Allyl radical, we have to place the three mobile electrons in orbits that embrace all three nuclei, and which, in the neighbourhood of any one of them, resemble one of the dumb-bell atomic orbits. If we call the nuclei 1, 2, 3 and write ψ_1, ψ_2, and ψ_3 for these dumb-bell orbitals, the molecular orbitals will, by analogy with the case of H_2 which we discussed earlier, be written

$$\psi = c_1\psi_1 + c_2\psi_2 + c_3\psi_3.$$

The constants c_1, c_2, and c_3 have to be chosen so that this wave function is as nearly as possible a solution of the wave equation. This can be done by a minimum energy method, and the analysis closely resembles that required in finding the normal modes of vibration of three particles on the three Carbon centres. It can be shown that there are three possible molecular orbits of the required character, and their energies may be calculated. If E_0 is the energy of an isolated $2p_z$ atomic orbit, these molecular energies are

$$E_0 + \sqrt{2}\beta, \quad E_0, \quad E_0 - \sqrt{2}\beta,$$

where β is a certain constant, with a negative value, whose magnitude may be obtained from comparison measurements on Ethane and Ethylene. The first of these orbits has the greatest binding energy, and we therefore allot two electrons to this level; the remaining electron goes into the middle level, so that the total energy of the mobile electrons is $3E_0 + 2\sqrt{2}\beta$. If we had supposed (incorrectly, as it now appears) that two of the electrons had formed a double bond between nuclei 1 and 2, while the third electron remained unpaired on nucleus 3 (diagram (*b*)), or, alternatively, that the double bond had been between 2 and 3 with the unpaired

electron localised on nucleus 1 (diagram (*c*)), the energy would have been $3E_0+2\beta$. Since β is numerically negative, there is thus an increase of binding energy due to the extra mobility of these electrons, above that to be expected on the basis of simple additivity of bond energies. We may describe the resulting bonds as *non-localised* bonds, thereby differentiating them completely from the previous *localised* bonds. The extra energy gained by removing the restriction to localisation has been called the *resonance energy*. Its value has been determined experimentally for many of these compounds. In the case of Allyl the resonance energy is about 15 kcals.

Another physical interpretation of this resonance energy is sometimes given as follows (although it does not fall strictly within the present line

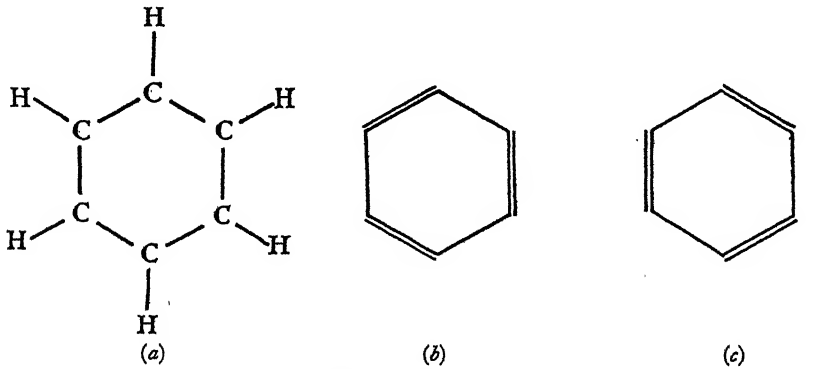


FIG. 9.—Benzene.

of thought): if we had tried to draw a chemical bond diagram for this molecule we should have been forced into either (*b*) or (*c*) of fig. 8. If we say that neither of these states represents the true picture of this molecule, but that there is very rapid resonance from one to the other and back, this resonance taking place so rapidly that it cannot be observed, then, on the basis of mechanical problems in which resonance occurs, we shall expect a lowering of the energy. In this case the lowering is about 15 kcals, and on account of the rapid changing about, the two bonds cannot be called either single or double bonds, but partake of the properties of both. This interpretation is the basis of the electron-pair, or valence-bond, approximation.

A similar, and extremely important, molecule is Benzene C_6H_6 (fig. 9 (*a*)). It is known from a variety of evidence that all the atoms, Hydrogen and Carbon, lie in a plane. One of the classic problems of organic chemistry has been to decide how the electrons are allotted to this molecule, and what kinds of bonds they will form. The solution that went furthest to explain the chemical behaviour of Benzene was that

of Kekulé, who supposed that instead of having either of the two possible configurations (*b*) and (*c*) in which single and double bonds alternate, the molecule was continually changing, oscillating from one structure to the other. But on our present viewpoint this can be explained somewhat differently. For when we come to allot the electrons, we shall first allot two electrons to each of the twelve "sausage" bonds, and we see that for this to be possible the Carbon atoms must each be in a trigonal state.

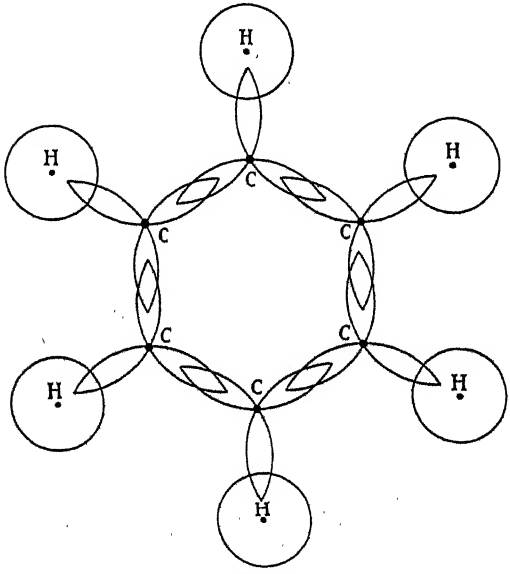
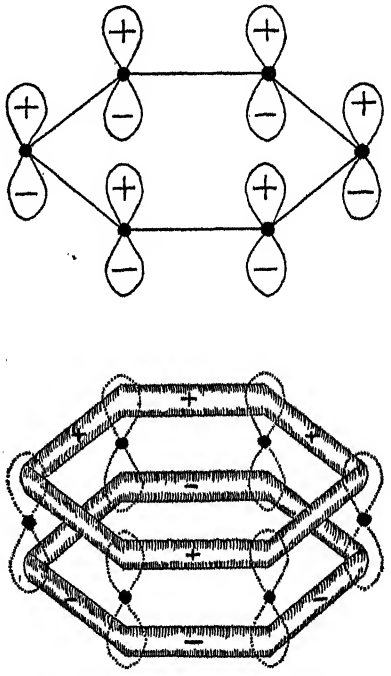


FIG. 10.—Overlapping of atomic σ-type orbits in Benzene.



Mobile electrons in Benzene C_6H_6.
FIG. 11.

The atomic orbits from which these "sausage" bonds are constructed are shown in fig. 10, which explains why this molecule and other aromatic molecules form plane rings containing six Carbon atoms. (In no other way can we get the required optimum angles of 120° for maximum overlapping from neighbouring atoms.) A simple counting of electrons shows that there are now six electrons left unpaired; these are the dumb-bell electrons, all perpendicular to the plane of the hexagon, and therefore parallel. If these electrons did not interact with each other to form molecular orbits, they would resemble the top picture of fig. 11. However, when we allow them to interact, they become free to migrate from one nucleus to another; they are mobile electrons like the mobile electrons of the Allyl radical, and they take up molecular orbits which we could

represent diagrammatically by the lower picture in fig. 11. The streamer bonds which we saw were typical of Ethylene are now spread out and we have two streamers going right round the molecule, above and below the central plane.

If we call the nuclei 1 . . . 6, then these molecular orbitals are represented by suitable sums of the atomic orbitals:

$$\psi = c_1\psi_1 + c_2\psi_2 + \ldots + c_6\psi_6.$$

The constants c_r must be chosen so that this function is as good a solution as possible of the wave equation. The necessary calculations are quite simple and it can be shown that there are six possible molecular levels, each with a different set of constants c_r, and corresponding energies E_r. If E_0 and β have the same meanings as before, the actual orbitals and energies are shown in the following table:—

$$\begin{aligned}
\Psi_1 &= \psi_1 + \psi_2 + \psi_3 + \psi_4 + \psi_5 + \psi_6, & E_1 &= E_0 + 2\beta \\
\Psi_2 &= \quad \psi_2 + \psi_3 - \psi_5 - \psi_6, & E_2 &= E_0 + \beta \\
\Psi_3 &= 2\psi_1 + \psi_2 - \psi_3 - 2\psi_4 - \psi_5 + \psi_6, & E_3 &= E_0 + \beta \\
\Psi_4 &= 2\psi_1 - \psi_2 - \psi_3 + 2\psi_4 - \psi_5 - \psi_6, & E_4 &= E_0 - \beta \\
\Psi_5 &= \quad \psi_2 - \psi_3 + \psi_5 - \psi_6, & E_5 &= E_0 - \beta \\
\Psi_6 &= \psi_1 - \psi_2 + \psi_3 - \psi_4 + \psi_5 - \psi_6, & E_6 &= E_0 - 2\beta
\end{aligned}$$

The first three of these levels may be called bonding orbits, since the energy is less than that of an isolated $2p_z$ atomic orbit (β negative). The other three are anti-bonding. In the normal state of Benzene there will be two electrons in each of the bonding orbits, so that their energy is $2(E_1+E_2+E_3) = 6E_0 + 8\beta$. Now the energy of a single Kekulé structure such as fig. 9 (*b*) or (*c*) is $6E_0 + 6\beta$, so that the resonance energy is 2β, *i.e.* about 30 kcals; this agrees quite well with the experimental value.

Confirmation of the point of view regarding Benzene can be obtained from two other independent sources. In the first place the occupied orbits Ψ_1, Ψ_2, and Ψ_3 embrace all six Carbon nuclei, and we may interpret this to mean that the electrons in these orbits can move in a "closed circuit" round the Benzene ring. In the absence of any magnetic field they are equally likely to travel in either direction round the ring, but in the presence of such a field there will be a preference for one rather than the other; thus there should be a large diamagnetism when the magnetic field is perpendicular to the plane of the molecule. This is, indeed, precisely what is found experimentally, and its magnitude is in accord with the predicted "electron current" round this small circuit. The second confirmation comes from the absorption spectrum. The characteristic absorption of Benzene in the ultra-violet may be explained

by supposing that one of the electrons in the molecular orbits $\Psi_1 - \Psi_3$ is excited into one of the unoccupied orbits $\Psi_4 - \Psi_6$. Explicit calculations show that the frequency of this absorption band is just what we should have expected it to be on this theory. It may be added here that the characteristic colour of many of the organic dyes is explained in precisely the same way, as being due to the excitation of a mobile electron from an occupied (and bonding) orbit to an unoccupied (and anti-bonding) orbit.

Similar analysis to that which we have described for Benzene can be carried through for other molecules of this nature. Naphthalene (fig. 12) $C_{10}H_8$ is an example; the nuclear framework is planar, and the ten mobile electrons which remain after allocating the localised σ-type bonds are free to move over all the ten Carbon nuclei. We should write for the wave function of a typical member of these mobile electrons

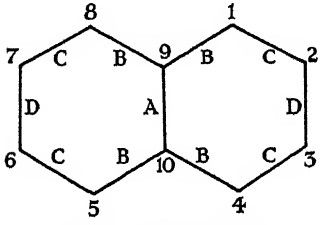


FIG. 12.—Naphthalene.

$$\psi = c_1\psi_1 + c_2\psi_2 + \ldots + c_{10}\psi_{10},$$

and choose the constants c_r so that this is as good a solution of the wave equation as possible. The calculation of the energies and the c_r involves the solution of a determinantal equation of the 10th degree, but this is not impossible, and the wave functions and energies can be determined. The calculated resonance energy is in tolerably good agreement with the experimental value.

The table below gives a few of the calculated and experimental values for the resonance energies of some of these unsaturated molecules. It can be seen that the agreement with experiment is reasonably good, in view of the nature of the approximations used.

RESONANCE ENERGIES (KCALS).

Molecule.	Resonance Energy. Calculated.	Resonance Energy. Experimental.
Ethylene	..	..
Butadiene	5	5
Hexatriene	11	..
Octatetraene	17	..
Benzene	30	37
Diphenyl	65	87
Naphthalene	63	75

We may use these calculations to study the way in which such unsaturated molecules are hydrogenated; for suppose that two atoms of Hydrogen are added on to the Naphthalene molecule. This will reduce

the number of mobile electrons, and alter the possible types of molecular orbits. The energy can be calculated for each of the 18 possible molecules that can thus be formed, and it appears that the 1,2 addition is energetically the most preferred. This is, indeed, what is found experimentally. The course of further successive additions of Hydrogen can be followed in a similar manner.

6. Bonds of Fractional Order.

There is a further interesting development in this particular field of mobile electrons. As a result of recent improvements in X-ray and electron-scattering apparatus, and in analysis of infra-red and Raman spectra, it has become possible to determine the lengths of the C–C bond in many molecules. Certain conclusions can be drawn.

For in cases where we have been accustomed to draw a single bond, and in which the Carbon atom is in the tetrahedral state which we explained earlier, Ethane C_2H_6 being the typical example, the C–C link is, almost within the limit of measurement, constant and equal to 1·54 A. In cases such as Ethylene C_2H_4, where we have an isolated double bond, the link is about 1·33 A., and in cases of a triple bond it is about 1·20 A. In fact, wherever we have localised electron orbits, and consequent additivity of bond energies, we also find definite values for the bond lengths. But in molecules such as those of § 5 (Benzene, Naphthalene, etc.), where there are non-localised or mobile electrons, the links have lengths which are neither those of a pure single nor of a pure double bond, but are intermediate between them; it is natural to describe them as links of fractional order, and we may draw a curve (fig. 13) of order plotted against length. This curve would enable us to fix the order if we knew the length, or, which is of more use in practice, would enable us to predict the length if we could calculate the order.

The explanation of these queer bonds of fractional order, which are neither single nor double, but intermediate between the two, must lie in the behaviour of the mobile electrons—for these electrons represent the only essential difference from the normal single and double bonds.

Let us split up the order of a bond, as for example in Benzene, into two parts: the one part, which is unity, arises from the single bonds which we saw were first established; if we call the remainder p, then p, which is generally a fraction between 0 and 1, will be the contribution to the order from the mobile electrons; we might even call it the *mobile order* of the bond. Then $p=0$ for pure single bonds, as Ethane;

$p=1$ for pure double bonds, as Ethylene; $p=2$ for pure triple bonds, as Acetylene.

Let us consider one of the mobile electrons; we have seen that its wave function is of the form

$$\psi = c_1\psi_1 + c_2\psi_2 + \ldots + c_n\psi_n,$$

where the Carbon nuclei are numbered from 1 to n, and ψ_1 is the dumb-bell orbit round nucleus 1, etc. This electron has an orbit which extends

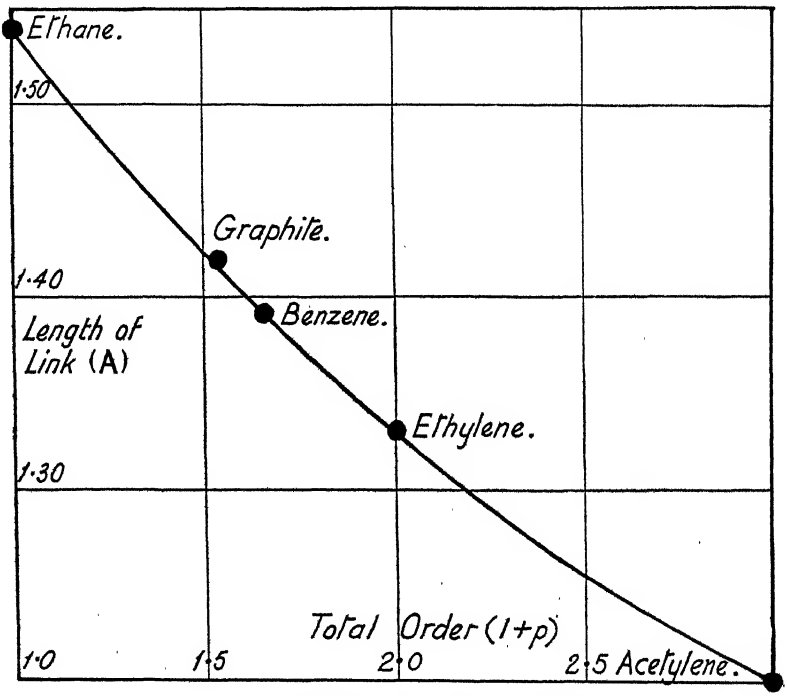


FIG. 13.—Order and length of a C–C bond.

over all the nuclei, and it may be thought of as spending a certain fraction of its time in the region between any two adjacent nuclei. If this fraction is large, we can say that this particular electron contributes considerably to the mobile order of the bond between these nuclei. We can make this more precise by saying that if our original wave function ψ is normalised, c_r^2 is the probability of finding the electron round nucleus r, and c_s^2 the probability of finding it around nucleus s. Unless both c_r and c_s are reasonably large, there is little chance of finding it between these two nuclei. Let us therefore define the contribution of this electron to the mobile order p between nuclei r and s as c_rc_s. This applies, of course, only to neighbour nuclei. The total mobile order of a bond is therefore

the sum of contributions $c_r c_s$ from each of the mobile electrons present: we may write it

$$p = \Sigma c_r c_s,$$

and the total order of the bond between r and s is $1 + \Sigma c_r c_s$.

It can easily be verified that this definition of order gives a correct value for Ethylene ($p = 1$) and Acetylene ($p = 2$). Further, it can be shown on this definition that the energy of a bond of mobile order p is p times a single-bond energy $+(1 - p)$ times a double-bond energy. Our theory therefore provides a reasonable justification for the use of the idea of fractional order.

It is not difficult now to calculate the order of any bond and thus, from the curve, to predict the length: for we have merely to determine the coefficients c_r in the wave function for each mobile electron.

When we make this calculation for Benzene we discover that all the links are equivalent, and $p = 2/3$, so that the total order is 1·67. Reading off from the curve gives the length as 1·39 A., which is exactly the experimental value. Similar calculations for the graphite crystal, which is like a huge molecule of this type, give an order 1·55 and a length 1·42 A., in agreement with the observed value.

The conclusion that all the bonds in Benzene are equivalent is very interesting because it resolves the problem of the two Kekulé structures; the molecule must not be thought of as being in either of the states usually drawn, but rather in an entirely different state in which single and double bonds do not appear but all the bonds are exactly equivalent, approximately two-thirds of the way between a pure single and a pure double bond.

Naphthalene (fig. 12) illustrates the same phenomenon. Using the notation shown in the figure, the orders and lengths of the links are calculated to be those in the following table:—

Link	A	B	C	D	Mean
Order	1·518	1·555	1·725	1·603	1·622
Length * (A)	1·417	1·402	1·371	1·400	1·394

It is seen that the bonds are all very much the same, though those labelled C (the 1–2 bonds) are most nearly double bonds. This result is in complete accord with the chemical experiments which show that the 1,2 position is very reactive. The detailed verification of the lengths in this molecule cannot be made, but the mean length agrees well with experiment, and the A link is believed, from X-ray analysis, to be somewhat longer than the others.

* The last decimal in this and other predicted lengths is important only for the purposes of comparison.

When two Benzene rings are bonded together, as in Diphenyl (fig. 14), the atoms all lie in one plane. It can be shown that the links in each Benzene ring are hardly affected in length, but the apparent single bond joining the two rings has an order 1·37 and a length 1·45 A. It is very far from being the straightforward single bond that it is usually drawn! The experimental value for this link, from X-ray analysis, is slightly higher, 1·48 A., but the accuracy of this latter value does not exceed 0·02 A.

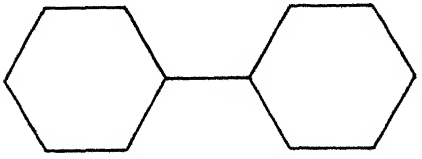


FIG. 14.—Diphenyl.

A series of interesting molecules is shown in fig. 15. They are all planar molecules, and the mobile electrons are able to migrate from one end to the other; this explains why a substitution in one Benzene ring will affect conditions of further substitution at the far end of the other

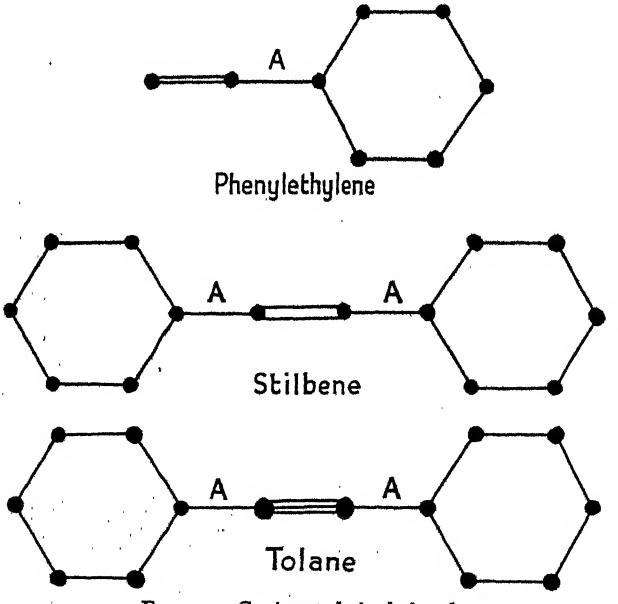


FIG. 15.—Conjugated single bonds.

ring. The interesting link in all these molecules is the link A, which is usually drawn as a single bond. However, it is seen from the table below that its length is about half-way between a pure single bond 1·54 A. and a pure double bond 1·33 A.

Molecule.	Length of Link A.
Phenylethylene	1·44 A.
Stilbene	1·44 A.
Tolane	1·40 A.

The lengths of these links have been measured by X-ray analysis, and agree with the calculated values to within 0·01 A.

Another interesting series of molecules is the conjugated chains $C_{2n}H_{2n+2}$, of which Ethylene is the first example. These molecules are usually written with alternating single and double bonds, starting at each end with a double bond. However, there is a variety of evidence, from optical rotation and the parachor, to show that the bonds are not so simple as this. If we suppose that the nuclei lie in a plane and form some kind of zigzag chain of which the angles are all 120°, as for example in octatetraene (fig. 16), then the mobile electrons are able to migrate over the whole molecule, and we may use our previous technique to describe the orders and lengths of the various bonds. The table below shows the lengths of the first few members of this series. It appears that the links do alternate in size, but that these alternations are much less marked away from the end, and the bonds all tend to a uniform length rather nearer to the double bond than to the single, as we move towards the centre of the molecule. None of these bonds have lengths corresponding to a pure single or a pure double bond. If we deal with the infinite conjugated chain it appears that the same process takes place, but that differences in length become inappreciable after about the seventh from each end.

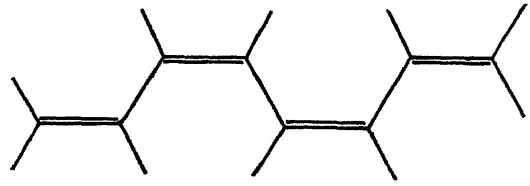


FIG. 16.—Octatetraene.

LENGTHS IN THE CONJUGATED CHAINS $C_{2n}H_{2n+2}$.

Molecule.	*n*.	Bond Diagram.	Link Numbers.
Ethylene	1	=	1
Butadiene	2	= – =	1, 2, 1
Hexatriene	3	= – = – =	1, 2, 3, 2, 1
Octatetraene	4	= – = – = – =	1, 2, 3, 4, 3, 2, 1

Link Number	1	2	3	4
Ethylene	1·331	..	..	..
Butadiene	1·345	1·432	..	..
Hexatriene	1·351	1·424	1·366	..
Octatetraene	1·353	1·422	1·371	1·415

In this section, and in the preceding section, we have discussed bonds of fractional order, or *non-localised* bonds, between pairs of Carbon atoms. But our analysis, and the general results, hold equally well for other atoms which form multiple bonds, such as Nitrogen, Oxygen, Sulphur, or Chlorine.

7. Other Applications.

There are many other applications of the methods that have just been described; we may refer briefly to three of them.

(a) *Rotation about a Conjugated Single Bond.*—When we discussed the Diphenyl molecule (fig. 14) in which a so-called single bond connected two Benzene rings, we found that it was not in fact a true single link at all, but had an order 1·37. This implies that it has certain characteristics of a double bond, and we should therefore expect that it would exert an influence tending to restrict any possible rotation about it. The magnitude of this effect, and a similar effect in all such molecules where a single bond is to be found between two double bonds, or Benzene rings, has not been calculated accurately; but an approximate value may be found as follows. If we rotate the two Benzene rings till their planes are at 90° to each other, the mobile electrons will not be able to move from one-half of the molecule to the other, because the directions of the relevant component dumb-bells are perpendicular. The loss of energy in this step is found to be about 7 kcals, and this would therefore be approximately the height of the potential barrier (or of that part of it which arises from these electrons) that resists such free rotation.

(b) *Vibrations.*—The second application is to vibration frequencies. It is possible to calculate how the energy of these unsaturated molecules depends upon the scale of the molecule; all that is needed is a knowledge of the manner in which the quantity β introduced in § 5 depends upon the separation of two neighbouring Carbon atoms, and this may be found from the vibration frequencies of Ethane and Ethylene.

We have seen in Benzene, for example, that the equilibrium configuration is a regular hexagon of side 1·39 A. Suppose that the whole molecule expands without change of shape; we may call this "breathing"; then the change in energy can be calculated and from this the frequency of these scale-model vibrations can be determined. The calculated value is 1038 cm.$^{-1}$, which compares excellently with the experimental value of 991 cm.$^{-1}$. It is not, however, yet possible to discuss in this way vibrations in which the angles of the hexagon change because this involves a distortion of the trigonal bonds in the plane of the hexagon.

Other vibrations could be studied in which the different links varied in a different manner. For example, in the Allyl radical CH_2–CH–CH_2, which we discussed earlier, it is possible to determine the energy of the molecule as a function of x_1 and x_2, the lengths of the two C–C links. The equilibrium configuration, with minimum energy, corresponds to

$x_1 = x_2 = 1{\cdot}37$ A., but it is possible from a knowledge of the energy contours to estimate the frequencies of those normal modes in which the angles at the central Carbon atom remain 120°.

(c) *Polymerisation.*—The third, and last, application of the theory of mobile electrons is to the question of polymerisation, a matter of considerable importance in the field of plastics and synthetic rubbers. Here there is interaction between mobile electrons of one molecule and another;

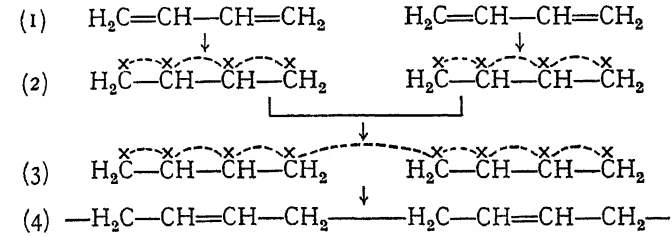


FIG. 17.—One possible polymerisation of Butadiene.

such interaction is possible if the molecules are arranged suitably relative to each other, and may perhaps form a temporary union by which other transformations take place. Thus in the example shown in fig. 17, between two Butadiene molecules, in (1) we show the conventional formulæ, and in (2) the diagrammatic representation by means of mobile electrons; in (3) there is a fusion of the two molecules, the mobile electrons forming resonance across the ends; and in (4) there is a rearrangement of the bonds and formation of a longer chain. In this way

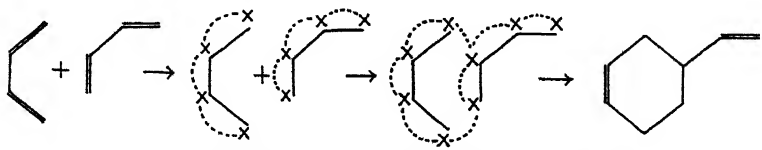


FIG. 18.—Another possible polymerisation of Butadiene.

the building up of huge molecules such as occurs in polymerisation may possibly take place. It is not necessary, however, for the interaction to take place across the ends of both reacting molecules, and it might, as is often supposed, equally well take place between other Carbon atoms of the two molecules, as in fig. 18, to form vinyl cyclohexene.

There are other effects which we might have outlined, such as dipole moments, electronic excitation, electric and magnetic susceptibilities, and preferential substitution; in part these are already explained, but there is still much work to be done, particularly from the quantitative point of view. Further progress will depend increasingly upon a fusion of interest between the theoretical and experimental worker.

SUMMARY.

Wave mechanics is able to describe with some precision the motions of electrons in atoms, but when we study molecules we have to use more approximate descriptions. It turns out that what the chemist is accustomed to call a single bond is in reality a pair of electrons, having opposed spins, describing equivalent orbits which have symmetry about the line joining the two nuclei concerned; this may be called a localised bond. The tetrahedral character of the bonds from saturated Carbon atoms are easily fitted into this scheme.

In Ethylene, however, another type of orbit appears; this is the double-streamer orbit, and two electrons in this orbit convert a normal single bond into a double bond. Again the bond is a localised bond, with a characteristic energy and length.

In more complex molecules, such as Benzene, there is a framework of single bonds, and the remaining electrons have orbits that embrace all six of the Carbon atoms; these mobile electrons give the aromatic and conjugated molecules their characteristic properties, but as a result the bonds are neither pure single bonds nor pure double bonds, but a hybrid of the two, and the electrons in these bonds are no longer localised in the region between any two particular nuclei. The energies of these molecules can be calculated in fair agreement with experiment, and from a knowledge of the wave function it is possible to define an order, which is usually fractional, for these bonds. In Benzene all the C-C links are equivalent, and their order is $1\frac{2}{3}$.

A curve which connects the fractional order with the length of the bond enables us to predict the lengths of these bonds, and, where experimental comparison is available, agreement is found.

These mobile electrons are important in a study of vibration frequencies, restricted rotation about C–C bonds, and in polymerisation.

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INDEX SLIP

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XII.—Some Remarks occasioned by the Geometry of the Veronese Surface.

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INTRODUCTION.

THE subject-matter of these pages may be briefly summarised as follows: the geometry of the Veronese surface, with an algebraic representation of it that does justice to its self-dual character; the relations of the secant planes of the surface to quadrics which either contain the surface or are outpolar to it; and the derivation of an invariant and two contravariants of a ternary quartic in the light of the (1, 1) correspondence between the quartic curves in a plane and the quadrics outpolar to a Veronese surface. There is no suggestion of discovering fresh properties of the surface, though possibly the results in § 12 and § 13 may be new; but the geometrical considerations lead naturally to some algebraical results which it seems worth while to have on record, such as, for example, the identity 8.2 and the remarks concerning the rank of the determinant which appears there, and the form found in § 13 for the harmonic envelope of a plane quartic curve. These algebraical results lie very close to properties of the surface; so close in fact that one might say that the Veronese surface is the proper *mise en scène* for them.

The introduction of the factor $\sqrt{2}$ into our equation 2.1 may seem a trivial matter; yet it is essential. The fountain-head of the subject is Veronese's paper of 1884, where the surface made its first appearance and as a result of which it was given its name; and the plain fact is that Veronese's algebra on pp. 355 and 356 is wrong, and wrong solely because of the omission of this normalising factor. Veronese recognised at first sight the symmetry and regularity of the configuration that is built upon the surface which geometers have named after him; he supposed, and rightly, that its self-dual nature would be shown too by the algebra if the co-ordinate system were properly chosen. And his geometrical insight enabled him to give the correct results without depending on any algebra to discover them, while he was so sure of the geometry that he must never have troubled to subject his algebra to any test. So he says, for

example, that the quadric $\Sigma x_i^2=0$ touches the surface given by (5) of his § 16 along a quartic curve; a statement manifestly untrue for his equation as it stands, but which would become true if his equation were replaced by our 6.1. The crux of the matter is that the surface which is given, as a locus of points, by (5) of Veronese's § 16, and the surface which is given, as an envelope of primes, by (1) of Veronese's § 18, are two different surfaces and not the same surface.

The parametric equations of a Veronese surface regarded, on the one hand, as a locus of points and, on the other hand, as an envelope of primes, ought to be symmetrical in the point and prime parameters; this is so with our equations 6.1 and 6.2. But Veronese only achieved symmetry at the expense of accuracy; while Bertini, in his textbook, is accurate at the expense of symmetry.

The classical textbook exposition of the properties of the Veronese surface is Bertini's; the self-dual character of the configuration is fully brought out in the geometrical sections of Bertini's chapter, but the algebra, though perfectly correct, does not attain the same balance. Perhaps it was the consciousness of this that caused Bertini to relegate the parametric form of the surface in prime co-ordinates, corresponding to our 6.2, to a footnote. In any event the impression derived from this otherwise excellent chapter cannot but be that the algebra is inadequate to do justice to the configuration. The slur thus cast upon the algebra is unmerited, and it is high time it was removed.

This paper falls into two parts, the first consisting of §§ 1–10 and the second of §§ 11–16.

A certain proportion of the first part is naturally concerned with setting up the configuration; the fact that the algebra has the desired symmetry seems to justify the writing of a few pages, from this aspect, about a configuration whose properties are already well known. When descriptive arguments are employed, as, for instance, in § 7 and § 10, it has been thought preferable to argue directly from the geometry of the configuration itself, and not to invoke properties of systems of conics in a plane, as has often been done in order to obtain properties of the configuration. In § 5 a condition is found for a secant plane of a Veronese surface to meet a quadric in two lines; this condition is used in § 8 and in § 13. The discriminant of any quadric which contains a Veronese surface is, when the surface is given parametrically in canonical form, a determinant which has appeared in other writings; references to these have caused the writing of §§ 9 and 10, § 9 consisting of some remarks upon a paper of Sylvester's.

It is in § 11 that the quadric which is associated with a plane quartic curve and is outpolar to the Veronese surface is introduced, and the

quartic curve and this associated quadric are the core of §§ 11–16. The results obtained for the quartic curve could be extended to primals of any even order in space of any number of dimensions, and the work has been written out with an eye to such extensions. But the paper is sufficiently long, so that matters have been left, for the time being, as they stand.

Let it be emphasised that any algebraic result which appears here does so in consequence of studying the geometry of the Veronese surface.

I.

1. A point of a plane σ may be identified by the ratios of three homogeneous co-ordinates x_1, x_2, x_3 and a line by the ratios of three homogeneous co-ordinates u_1, u_2, u_3. The equation

$$u_1x_1 + u_2x_2 + u_3x_3 = 0 \tag{1.1}$$

may then be interpreted in either of two ways; it may imply that the point whose co-ordinates x_1, x_2, x_3 vary subject to 1.1 lies on the fixed line (u_1, u_2, u_3), or else that the line whose co-ordinates u_1, u_2, u_3 vary subject to 1.1 passes through the fixed point $\{x_1, x_2, x_3\}$. It is usual to denote a line by the *row-vector* u formed by its co-ordinates written in their proper order,

$$u = (u_1, u_2, u_3);$$

while a point is denoted by the *column-vector* formed by its co-ordinates written in their proper order,

$$x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = (x_1, x_2, x_3)' = \{x_1, x_2, x_3\},$$

the dash indicating transposition of row into column, an operation sometimes indicated, in order to save vertical space, by writing the co-ordinates in a horizontal row but enclosing them in the brackets $\{\}$. The ordinary row into column rule for multiplication of matrices then gives

$$ux = u_1x_1 + u_2x_2 + u_3x_3.$$

2. Now take the squares and products of the three co-ordinates x_i and write, τ being a square root of 2,

$$x^{[2]} = \{x_1^2, x_2^2, x_3^2, \tau x_2x_3, \tau x_3x_1, \tau x_1x_2\}. \tag{2.1}$$

As the point x varies in σ the point $x^{[2]}$, having six homogeneous co-ordinates, varies in a five-dimensional space Σ; it is, however, constrained to lie on a certain surface whose points are in (1, 1) correspondence with those of σ, a point of the surface and a point of σ corresponding to one another when they have the same values for the mutual ratios of the x_i.

More generally we may take the surface F in Σ such that the co-ordinates of a point of F are the six constituents $y_1, y_2, y_3, y_4, y_5, y_6$ of the column-vector

$$y = \mathrm{M}x^{[2]}, \tag{2.2}$$

where M is any non-singular matrix of six rows and columns; it is presumed to be non-singular because otherwise there would be a linear relation $vy = 0$ connecting the constituents of y, and the surface would then lie in a prime, or four-dimensional space, of Σ.

Any prime $vy = 0$ meets F in a curve whose points correspond to those of the conic $v\mathrm{M}x^{[2]} = 0$; since two conics in σ have four intersections there are four points common to F and two primes. Thus F is a quartic surface, having its prime sections represented by the conics of σ; it is the well-known Veronese surface, whose properties are classical (Veronese, 1884; Segre, 1885; Bertini, 1923).

3. The lines $ux = 0$ of σ correspond to a set of curves on F which, by analogy with the lines of σ, have the properties that any two of the curves have one common point, while through any two points of F there passes one and only one of the curves. Since a line $ux = 0$ and a conic $v\mathrm{M}x^{[2]} = 0$ in σ have two intersections, the curves on F are met by the prime sections in two points, and so are conics. The doubly infinite set of planes in which these conics lie will be called the *secant planes* of F.

If $ux = 0$ and $y = \mathrm{M}x^{[2]}$, then

$$\begin{array}{lllllll} (u_1 & . & . & . & \tau^{-1}u_3 & \tau^{-1}u_2)\mathrm{M}^{-1}y = 0, \\ (\,. & u_2 & . & \tau^{-1}u_3 & . & \tau^{-1}u_1)\mathrm{M}^{-1}y = 0, \\ (\,. & . & u_3 & \tau^{-1}u_2 & \tau^{-1}u_1 & .\,)\mathrm{M}^{-1}y = 0. \end{array} \tag{3.1}$$

All points of that conic on F which corresponds to points of σ for which $ux = 0$ satisfy these three linear equations, which are therefore the equations of primes containing the secant plane and so, being linearly independent, determine this secant plane.

A prime $vy = 0$ meets F, in general, in a rational quartic curve corresponding to a conic of σ. But if the conic of σ is the line $ux = 0$ taken twice, v must be such that the result of substituting the value of y given by 2.2 in vy is a constant multiple of the square of ux. We may suppose, absorbing this constant multiple into v, that vy is equal to the square of ux, and this is so when, and only when

$$v = (u_1^2, u_2^2, u_3^2, \tau u_2u_3, \tau u_3u_1, \tau u_1u_2)\mathrm{M}^{-1} = u^{[2]}\mathrm{M}^{-1}. \tag{3.2}$$

These primes will be called the *tangent primes* of F; each of them meets F in one of its conics taken twice, and 3.2 is a parametric representation of

them just as 2.2 was of the points of F. It is these primes that are the "spazi tangenti doppi" of Veronese and Segre.

We have considered points of F for which x satisfies 1.1 with fixed u; it is natural to consider also tangent primes of F for which u satisfies 1.1 with fixed x. If $ux=0$ and $v=u^{[2]}M^{-1}$, then

$$\begin{aligned} vM\{x_1 \quad . \quad . \quad . \quad \tau^{-1}x_3 \quad \tau^{-1}x_2\}&=0,\\ vM\{\,. \quad x_2 \quad . \quad \tau^{-1}x_3 \quad . \quad \tau^{-1}x_1\}&=0,\\ vM\{\,. \quad . \quad x_3 \quad \tau^{-1}x_2 \quad \tau^{-1}x_1 \quad . \;\}&=0, \end{aligned} \tag{3.3}$$

three equations having an interpretation dual to that of the equations 3.1; they are linearly independent and so determine a plane, this plane being common to the primes which touch F along the respective conics which pass through the fixed point x of F. Now this plane is the one which contains the three points given by

$$\begin{aligned} y=M\{x_1 \quad . \quad . \quad . \quad \tau^{-1}x_3 \quad \tau^{-1}x_2\},\\ y=M\{\,. \quad x_2 \quad . \quad \tau^{-1}x_3 \quad . \quad \tau^{-1}x_1\},\\ y=M\{\,. \quad . \quad x_3 \quad \tau^{-1}x_2 \quad \tau^{-1}x_1 \quad . \;\}, \end{aligned} \tag{3.4}$$

and these, being obtained by the polarisations of 2.2, all lie in the tangent plane of F at x (*cf.* Edge, 1938, p. 473), which is therefore the plane determined by 3.3.

4. The tangent plane at a point of F is determined by the three points 3.4, while that at a second point of F is determined by the three points

$$\begin{aligned} y=M\{\xi_1 \quad . \quad . \quad . \quad \tau^{-1}\xi_3 \quad \tau^{-1}\xi_2\},\\ y=M\{\,. \quad \xi_2 \quad . \quad \tau^{-1}\xi_3 \quad . \quad \tau^{-1}\xi_1\},\\ y=M\{\,. \quad . \quad \xi_3 \quad \tau^{-1}\xi_2 \quad \tau^{-1}\xi_1 \quad . \;\}. \end{aligned} \tag{4.1}$$

The point

$$y=M\{x_1\xi_1,\ x_2\xi_2,\ x_3\xi_3,\ \tau^{-1}(x_2\xi_3+x_3\xi_2),\ \tau^{-1}(x_3\xi_1+x_1\xi_3),\ \tau^{-1}(x_1\xi_2+x_2\xi_1)\} \tag{4.2}$$

is manifestly linearly dependent both on the points 3.4 and on the points 4.1; it therefore belongs to both the tangent planes. Whence, although two planes of Σ do not meet in general, *every pair of tangent planes of F has a point of intersection.* Now the determinant

$$\begin{vmatrix} \tau x_1\xi_1 & \tau^{-1}(x_1\xi_2+x_2\xi_1) & \tau^{-1}(x_3\xi_1+x_1\xi_3)\\ \tau^{-1}(x_1\xi_2+x_2\xi_1) & \tau x_2\xi_2 & \tau^{-1}(x_2\xi_3+x_3\xi_2)\\ \tau^{-1}(x_3\xi_1+x_1\xi_3) & \tau^{-1}(x_2\xi_3+x_3\xi_2) & \tau x_3\xi_3 \end{vmatrix}, \tag{4.3}$$

being a numerical multiple of the discriminant of

$$(u_1x_1+u_2x_2+u_3x_3)(u_1\xi_1+u_2\xi_2+u_3\xi_3), \tag{4.4}$$

must vanish identically; hence the equation of the locus of the intersection of pairs of tangent planes of F is

$$\begin{vmatrix} \tau\gamma_1 & \gamma_6 & \gamma_5 \\ \gamma_6 & \tau\gamma_2 & \gamma_4 \\ \gamma_5 & \gamma_4 & \tau\gamma_3 \end{vmatrix} = 0, \tag{4.5}$$

where the γ_i are the six constituents of the column vector $M^{-1}y$. The locus is therefore a cubic primal C.

If the point ξ were to coincide with the point x, then 4.2 would be the point $Mx^{[2]}$ and so a point of F. But then 4.4 would be a perfect square, so that every first minor of 4.3 would vanish. Hence F is a double surface on C. Every chord of F must therefore lie on C as meeting it in at least four points, from which it follows that all the secant planes of F lie on C and also, since the chords of F include its tangents, that all the tangent planes of F lie on C.

It is the secant planes of F that afford the projective generation of C indicated by the determinant 4.5; this determinant is in fact obtainable at once by eliminating u_1, u_2, u_3 from 3.1.

5. Let Ψ be any quadric of Σ; its equation is

$$y'Ay = 0,$$

where A is a symmetric matrix, of six rows and columns. Every plane of Σ meets Ψ in a conic, and for certain planes of Σ this conic is a pair of lines. The condition to which the plane must be subjected in order that the conic should thus degenerate is obtained (*cf.* Bertini, 1923, p. 149), in the standard fashion, by bordering A; below by three rows arising from the equations of three linearly independent primes through the plane, to the right by the three columns obtained by transposition of these rows. In particular it follows, on referring to 3.1, that a secant plane of F meets Ψ in a pair of lines provided that the determinant of the matrix

$$\begin{pmatrix} A & M'^{-1}U' \\ UM^{-1} & . \end{pmatrix}$$

vanishes, where

$$U = \begin{pmatrix} u_1 & . & . & . & \tau^{-1}u_3 & \tau^{-1}u_2 \\ . & u_2 & . & \tau^{-1}u_3 & . & \tau^{-1}u_1 \\ . & . & u_3 & \tau^{-1}u_2 & \tau^{-1}u_1 & . \end{pmatrix}.$$

Since M is non-singular this matrix is equal to

$$\begin{pmatrix} M'^{-1} & . \\ . & I \end{pmatrix}\begin{pmatrix} M'AM & U' \\ U & . \end{pmatrix}\begin{pmatrix} M^{-1} & . \\ . & I \end{pmatrix},$$

where I is the unit matrix of three rows and columns; so that the condition is

$$\begin{vmatrix} M'AM & U' \\ U & . \end{vmatrix} = 0.$$

Those secant planes which meet Ψ in pairs of lines thus correspond to lines belonging to an envelope in σ of the sixth class.

6. Now let us arrange, as we always may, that M, through the agency of a collineation, is replaced by the unit matrix; then the points of F are given by

$$y = x^{[2]} \tag{6.1}$$

and the tangent primes by

$$v = u^{[2]}. \tag{6.2}$$

The point of intersection of two tangent planes of F is now

$$y = \{x_1\xi_1,\ x_2\xi_2,\ x_3\xi_3,\ \ \tau^{-1}(x_2\xi_3 + x_3\xi_2),\ \ \tau^{-1}(x_3\xi_1 + x_1\xi_3),\ \ \tau^{-1}(x_1\xi_2 + x_2\xi_1)\}, \tag{6.3}$$

while the equation of C is

$$\begin{vmatrix} \tau y_1 & y_6 & y_5 \\ y_6 & \tau y_2 & y_4 \\ y_5 & y_4 & \tau y_3 \end{vmatrix} = 0. \tag{6.4}$$

The first polars of the points of Σ with respect to C form a linear system of quadric primals of freedom 5; since F is a double surface on C all these quadrics contain F. Conversely: every quadric which contains F is the polar quadric of some point of Σ with respect to C. For any quadric whose equation is satisfied identically by the co-ordinates 6.1 must be a linear combination of the six quadrics

$$y_4^2 = 2y_2y_3 \qquad y_5^2 = 2y_3y_1 \qquad y_6^2 = 2y_1y_2$$
$$y_5y_6 = \tau y_1y_4 \qquad y_6y_4 = \tau y_2y_5 \qquad y_4y_5 = \tau y_3y_6,$$

while the polar quadric of the point $y = \eta$ with respect to C is

$$\eta_1(2y_2y_3 - y_4^2) + \eta_2(2y_3y_1 - y_5^2) + \eta_3(2y_1y_2 - y_6^2) + \eta_4(\tau y_5y_6 - 2y_1y_4) + \eta_5(\tau y_6y_4 - 2y_2y_5) + \eta_6(\tau y_4y_5 - 2y_3y_6) = 0. \tag{6.5}$$

7. A quadric which contains F, and so is a polar quadric of C, will be denoted by the symbol Q. Take any point of C which lies on Q and not on F; there is a secant plane of F passing through this point, and so meeting Q in the point as well as in a conic of F; the plane therefore lies entirely on Q. Hence the intersection of Q and C consists wholly of secant planes of F, and the sextic threefold K_3^6 common to Q and C is generated by a singly infinite family of secant planes.

The conics in which F is met by these planes of K_3^6 have an envelope

which, we prove, is the section of F by the prime which is the second polar, with respect to C, of that point of which Q is the first polar.

Let P be any point of F; the tangent cone of C at P is * the quadric cone Γ by which F is projected from T, its tangent plane at P, and the tangent primes of Γ are those tangent primes of F which pass through T. The polar quadric Q of any point A passes through P, and its tangent prime at P is † the polar prime of A with respect to Γ; this prime passes through T and so meets F in a pair of conics, these conics being those two which pass through P and have their planes on K_3^6. P is on the envelope of the conics when those two conics which pass through P and have their planes on K_3^6 coincide; in order that this should happen the tangent prime of Q at P must also be a tangent prime of F, and so of Γ. Wherefore the polar prime of A with respect to Γ must be a tangent prime of Γ, and A must lie on Γ. But Γ, the tangent cone of C at its node P, is the first polar of P with respect to C, so that the first polar of P must pass through A. Hence the second polar of A must pass through P, which is what we desired to prove.

Those secant planes of F which lie on Q are thus identified. If the equation of Q is 6.5, then the secant planes which lie on Q are those meeting F in conics which touch the section of F by the polar prime of $y=\eta$ with respect to Q. The equation of this polar prime is

$$y_1(2\eta_2\eta_3-\eta_4^2)+y_2(2\eta_3\eta_1-\eta_5^2)+y_3(2\eta_1\eta_2-\eta_6^2)$$
$$+y_4(\tau\eta_5\eta_6-2\eta_1\eta_4)+y_5(\tau\eta_6\eta_4-2\eta_2\eta_5)+y_6(\tau\eta_4\eta_5-2\eta_3\eta_6)=0. \quad (7.1)$$

The section of F by this prime clearly corresponds to the conic in σ whose equation is

$$\begin{vmatrix} \tau\eta_1 & \eta_6 & \eta_5 & x_1 \\ \eta_6 & \tau\eta_2 & \eta_4 & x_2 \\ \eta_5 & \eta_4 & \tau\eta_3 & x_3 \\ x_1 & x_2 & x_3 & 0 \end{vmatrix} = 0, \quad (7.2)$$

and whose line-equation is therefore

$$\eta_1u_1^2+\eta_2u_2^2+\eta_3u_3^2+\tau\eta_4u_2u_3+\tau\eta_5u_3u_1+\tau\eta_6u_1u_2=0; \quad (7.3)$$

the lines belonging to this envelope in σ correspond to those conics on F whose planes lie on Q.

* The cone must have the tangent plane of the double surface of C for its vertex, and must contain every line joining P to another point of the double surface.

† The tangent lines of the first polar of a point P with respect to a primal, at any point which is common to the polar and the primal, generate the locus which is the first polar of P with respect to the locus generated by the tangent lines of the primal at the point.

8. The matrix of the quadratic form on the left of 6.5 is

$$\mathbf{H} \equiv \begin{pmatrix} . & \eta_3 & \eta_2 & -\eta_4 & . & . \\ \eta_3 & . & \eta_1 & . & -\eta_5 & . \\ \eta_2 & \eta_1 & . & . & . & -\eta_6 \\ -\eta_4 & . & . & -\eta_1 & \tau^{-1}\eta_6 & \tau^{-1}\eta_5 \\ . & -\eta_5 & . & \tau^{-1}\eta_6 & -\eta_2 & \tau^{-1}\eta_4 \\ . & . & -\eta_6 & \tau^{-1}\eta_5 & \tau^{-1}\eta_4 & -\eta_3 \end{pmatrix}; \qquad (8.1)$$

it is the most general type of matrix for which $x'^{[2]}\mathbf{H}x^{[2]}$ is identically zero.

If $\mathbf{H}$ is now bordered, in the manner of § 5, by the rows U and the columns U′, the resulting determinant, when equated to zero, gives the condition for a secant plane of F to meet Q in a pair of lines. But every secant plane of F meets Q in a *non-degenerate* conic, namely, the conic in which the plane meets F; thus if the secant plane is such that the determinant vanishes it must be one of those which lie entirely on Q. But it has just been seen that those secant planes of F which lie on Q are those which satisfy 7.3, and no others; the inescapable conclusion is that, for a quadric Q, the sextic envelope mentioned at the end of § 5 must be the cube of the conic 7.3. The determinant must therefore be a numerical multiple of the cube of the left-hand side of 7.3; what precise multiple it is is seen at once by comparing the coefficients of $\eta_1{}^3u_1{}^6$.

Thus we have established the identity

$$\begin{vmatrix} . & \eta_3 & \eta_2 & -\eta_4 & . & . & u_1 & . & . \\ \eta_3 & . & \eta_1 & . & -\eta_5 & . & . & u_2 & . \\ \eta_2 & \eta_1 & . & . & . & -\eta_6 & . & . & u_3 \\ -\eta_4 & . & . & -\eta_1 & \tau^{-1}\eta_6 & \tau^{-1}\eta_5 & . & \tau^{-1}u_3 & \tau^{-1}u_2 \\ . & -\eta_5 & . & \tau^{-1}\eta_6 & -\eta_2 & \tau^{-1}\eta_4 & \tau^{-1}u_3 & . & \tau^{-1}u_1 \\ . & . & -\eta_6 & \tau^{-1}\eta_5 & \tau^{-1}\eta_4 & -\eta_3 & \tau^{-1}u_2 & \tau^{-1}u_1 & . \\ u_1 & . & . & . & \tau^{-1}u_3 & \tau^{-1}u_2 & . & . & . \\ . & u_2 & . & \tau^{-1}u_3 & . & \tau^{-1}u_1 & . & . & . \\ . & . & u_3 & \tau^{-1}u_2 & \tau^{-1}u_1 & . & . & . & . \end{vmatrix} \qquad (8.2)$$

$$\equiv \tfrac{1}{4}(\eta_1u_1{}^2 + \eta_2u_2{}^2 + \eta_3u_3{}^2 + \tau\eta_4u_2u_3 + \tau\eta_5u_3u_1 + \tau\eta_6u_1u_2)^3.$$

There is, however, more to be said about the determinant on the left of 8.2 than the mere identity alone tells us; for the secant plane of F not merely meets Q in a pair of lines, but lies entirely on Q. A reference to any of the textbooks, where the condition for a linear space to touch a quadric is obtained as the vanishing of a bordered determinant, will show that the argument employed can be extended to prove that the necessary and sufficient conditions for the space to meet the quadric in a cone with

an $[a]$ for vertex are that the rank of the determinant should fall below its full rank by $a+1$; ordinary contact, such as a plane meeting a quadric in two lines, corresponds to the case of $a=0$. Hence, for a secant plane of F which lies on Q, the rank of the determinant in 8.2 falls to 6, and the expression which appears cubed on the right is a factor of every seven-rowed minor.

9. The determinant $|\mathbf{H}|$ made its first appearance precisely a century ago (Sylvester, 1841, p. 232), in a slightly different form without the factors τ. It is not written down by Sylvester as a determinant, but the reader is left to observe its structure from the coefficients in six equations. Later, in 1856, Cayley encountered it again and observed that it was a numerical multiple of the square of the determinant

$$|h| = \begin{vmatrix} \tau\eta_1 & \eta_6 & \eta_5 \\ \eta_6 & \tau\eta_2 & \eta_4 \\ \eta_5 & \eta_4 & \tau\eta_3 \end{vmatrix}.$$

Cayley says that it would be desirable to have an *a priori* demonstration that $|h|$ occurs as a *squared* factor in $|\mathbf{H}|$; the natural setting for such a demonstration is the geometry of the Veronese surface, but before speaking of this it will be fitting to pass one or two remarks on Sylvester's paper, which is historic for the reason that it is there shown for the first time how the resultant of three conics may be written down.

Sylvester, before giving the process for obtaining the resultant of any three conics, gives three particular examples; it is in the first of these that $|\mathbf{H}|$ occurs. In the present notation, Sylvester's proposal is to obtain the resultant of the three equations

$$\left.\begin{aligned} \eta_3 x_2^2 + \eta_2 x_3^2 - \tau\eta_4 x_2 x_3 = 0, \\ \eta_1 x_3^2 + \eta_3 x_1^2 - \tau\eta_5 x_3 x_1 = 0, \\ \eta_2 x_1^2 + \eta_1 x_2^2 - \tau\eta_6 x_1 x_2 = 0. \end{aligned}\right\} \tag{9.1}$$

The coefficients of the squares of the variables were chosen in this way so that the dialytic process of elimination might the more easily be applied.

If the three equations are multiplied in order by x_1^2, $-x_2^2$, $-x_3^2$ and added, the result being then divided by $\tau x_2 x_3$, and if the corresponding processes derived by cyclically permuting the suffixes are also carried out, the three equations

$$\left.\begin{aligned} -\eta_4 x_1^2 - \tau\eta_1 x_2 x_3 + \eta_6 x_3 x_1 + \eta_5 x_1 x_2 = 0, \\ -\eta_5 x_2^2 - \tau\eta_2 x_3 x_1 + \eta_4 x_1 x_2 + \eta_6 x_2 x_3 = 0, \\ -\eta_6 x_3^2 - \tau\eta_3 x_1 x_2 + \eta_5 x_2 x_3 + \eta_4 x_3 x_1 = 0, \end{aligned}\right\} \tag{9.2}$$

are obtained.

The conics represented by these equations do not belong to the net 9.1; but, from the way in which they have been derived, they all pass through any base point the net 9.1 may have; *unless* such a base point lies on a side of the triangle of reference—a proviso which Sylvester, in his enthusiasm for the application of his new process of dialysis, overlooked. Thus, apart from the proviso, the necessary condition for the net 9.1 to have a base point is obtained by the elimination of

$$x_1^2, \quad x_2^2, \quad x_3^2, \quad \tau x_2x_3, \quad \tau x_3x_1, \quad \tau x_1x_2$$

between the six equations 9.1 and 9.2; the result of this elimination is precisely

$$|\,\mathbf{H}\,| = 0.$$

Sylvester then states incorrectly the value of $|\,\mathbf{H}\,|$; Muir says that this blunder is unaccountable, but the reason for it is the neglect of the proviso just mentioned. A glance at 9.1 shows that, if η_1 were to vanish, x_1 would be a factor of the left-hand sides of the second and third of the three equations, which would then certainly have a common solution. Thus η_1 must occur as a factor in the resultant, and so, by parity of reasoning, must η_2 and η_3. Knowing that the resultant must include the factor $\eta_1\eta_2\eta_3$, Sylvester jumped to the conclusion that this factor must also occur in $|\,\mathbf{H}\,|$; presumably he believed that the value of $|\,\mathbf{H}\,|$ was $\eta_1\eta_2\eta_3\,|\,h\,|$, and his wrong form for $|\,h\,|$ is a slip. He was, of course, perfectly correct in assuming that $\eta_1\eta_2\eta_3$ occurs as a factor of the resultant; but $|\,\mathbf{H}\,|$ is only a part of this resultant. The resultant of three conics is of degree four in the coefficients of each of them, as indeed appears at the end of this very paper of Sylvester's; thus the resultant of the equations 9.1 is of degree 12 in the η's, whereas $|\,\mathbf{H}\,|$ is only of degree 6. The resultant is actually $\eta_1^2\eta_2^2\eta_3^2\,|\,h\,|^2$, while $|\,\mathbf{H}\,| = -\frac{1}{4}\,|\,h\,|^2$.

The fact that their resultant is a perfect square must be a consequence of some peculiarity of the equations 9.1, and it is interesting that, in selecting the coefficients in 9.1 so that further quadratic forms 9.2 could be found to combine with them, Sylvester hit on a set of equations whose peculiarity is best explained in geometrical terms. For each of the three equations represents a pair of lines, and the six lines constituted by the three line-pairs are all tangents of the same conic; indeed the equations are those of the pairs of tangents from the vertices of the triangle of reference to the conic 7.3. Thus if the three line-pairs have a common point O three lines of the conic-envelope 7.3 pass through O; this envelope must then consist of two points O, O′ and the remaining three lines, one of each pair, pass through O′. Thus the peculiarity of the equations 9.1 is that if they have one common solution they have two; this is the reason why their resultant

is a perfect square. Incidentally the condition for 7.3 to be a point-pair is $|h| = 0$.

10. The determinant $|\mathbf{H}|$ was also encountered by Pasch (Pasch, 1891, p. 46), who constructed it as the Hessian of $|h|$ regarded as a cubic form in six variables. The geometrical interpretation of this in the space Σ is that, $|h| = 0$ being the equation of C, $|\mathbf{H}| = 0$ is the equation of that sextic primal which is the Hessian of C. We now give a geometrical proof that the Hessian of C is simply C itself, taken twice over.

The Hessian of a primal, in space of any number of dimensions, is the locus of those points whose polar quadrics with respect to the primal are cones; if a point is such that its polar quadric is a cone with a line-vertex the co-ordinates of the point cause all the first minors of the Hessian matrix to vanish, so that the point is a double point of the Hessian. Our object will thus have been achieved if it can be shown that the polar quadric of any point on C is a cone with a line-vertex; for it will then follow that every point of C is a node of its Hessian and therefore that the Hessian, since it is of the sixth order, consists of C taken twice over.

That the polar quadric of a point of C is a line-cone was proved by Segre (1885, p. 497); an alternative proof is as follows.

Let A be a point of C. The polar quadric Q_A is the first polar of A with respect to C; it has the same tangent prime as C at A and contains all those lines which pass through A and lie on C. Thus it contains those two tangent planes of F which pass through A. Let T_1, T_2 be the points of contact of these planes with F.

Any secant plane which passes through T_1 is met by Q_A both in the conic of F lying in the plane and in the line of intersection of the plane with the tangent plane of F at T_1; hence the plane lies entirely on Q_A. This applies to every secant plane which passes through T_1, so that all the chords which join T_1 to the other points of F lie on Q_A. Wherefore, since these chords do not all lie in a prime (for if they did F would do so too), Q_A must have T_1 as a node. Similarly, it has T_2 as a node, and so must be a cone with the line T_1T_2 for vertex. Hence *the polar quadric with respect to C of any point A of C itself is a line-cone*, the vertex of the line-cone lying on C and joining the two points where F is touched by those two of its tangent planes which pass through A.

The preceding argument shows, first, that $|\mathbf{H}|$ must be a numerical multiple of the square of $|h|$ and, secondly, that, as was noticed by Pasch, the rank of $\mathbf{H}$ falls to 4 whenever $|h| = 0$.

Incidentally, when $|h| = 0$ the envelope 7.3 consists of a pair of points; hence the locus K_3^6 generated by the secant planes of F which lie on Q_A must consist of two distinct cubic primals. It is clear, from the

descriptive argument above, that it does indeed consist of those cubic cones which project F from T_1 and T_2 respectively.

II.

11. Let $\phi(x_1, x_2, x_3)$ be a homogeneous quartic polynomial in x_1, x_2, x_3. Take the six expressions

$$\frac{\partial^2\phi}{\partial {x_1}^2}, \quad \frac{\partial^2\phi}{\partial {x_2}^2}, \quad \frac{\partial^2\phi}{\partial {x_3}^2}, \quad \tau\frac{\partial^2\phi}{\partial x_2 \partial x_3}, \quad \tau\frac{\partial^2\phi}{\partial x_3 \partial x_1}, \quad \tau\frac{\partial^2\phi}{\partial x_1 \partial x_2};$$

these are all homogeneous quadratic polynomials, and so are linear combinations of the six constituents

$${x_1}^2, \quad {x_2}^2, \quad {x_3}^2, \quad \tau x_2 x_3, \quad \tau x_3 x_1, \quad \tau x_1 x_2$$

of $x^{[2]}$. Arrange the six quadratic polynomials, in this order, vertically beneath one another; the coefficients of the constituents of $x^{[2]}$ then form a symmetric matrix, which is in fact

$$12\begin{pmatrix} a & h & g & \tau\alpha & \tau\eta' & \tau\gamma' \\ h & b & f & \tau\beta & \tau\zeta' & \tau\alpha' \\ g & f & c & \tau\gamma' & \tau\eta & \tau\beta' \\ \tau\alpha & \tau\beta & \tau\gamma' & 2f & 2\gamma & 2\beta \\ \tau\eta' & \tau\zeta' & \tau\eta & 2\gamma & 2g & 2\alpha \\ \tau\gamma' & \tau\alpha' & \tau\beta' & 2\beta & 2\alpha & 2h \end{pmatrix} = 12\mu, \text{ say,} \qquad (11.1)$$

where again $\tau^2 = 2$, and where the other fifteen letters denote the coefficients of the terms

$$\frac{4!}{i!j!k!} {x_1}^i {x_2}^j {x_3}^k, \quad \text{where} \quad i+j+k=4,$$

in ϕ. Then

$$\phi \equiv x^{[2]} \mu x^{[2]}.$$

The equation $\phi = 0$ represents a quartic curve δ, and, in the (1, 1) correspondence between the points of σ and those of F, δ corresponds to that curve Δ which is the intersection of F with the quadric $y\mu y = 0$. But, while Δ is uniquely determined when δ is given, there are other quadrics passing through it, every quadric $y(\mu + \mathbf{H})y = 0$ containing Δ for every $\mathbf{H}$ of the form 8.1; the two equations

$$x^{[2]}(\mu + \mathbf{H})x^{[2]} = 0, \qquad x^{[2]}\mu x^{[2]} = 0$$

represent the same curve δ because $x^{[2]}\mathbf{H}x^{[2]}$ is identically zero.

The quadric $y\mu y = 0$, whose matrix is the same as that formed by the

coefficients in the second polars of ϕ, has, however, a special property; it can thereby be identified, among the quadrics passing through Δ, without the equation $\phi=0$ being previously given, and without any reference to σ. For the quadrics which pass through Δ form a linear system of freedom 6, and if a quadric of this system is subjected to the six linear conditions of being outpolar to all those quadrics which are inscribed in F (*i.e.* which touch all the tangent primes of F), a unique quadric is thereby obtained, which is precisely the quadric $\dot{y}\mu y=0$. This result is due to Segre (Segre, 1892, p. 240); it is easily obtained by appealing to the special forms of the matrices $\mathbf{H}$ and μ. For, the parametric form for the tangent primes of F being given by 6.2, the matrix of the quadratic form in the prime coordinates u which corresponds to any quadric inscribed in F must be of the form $\mathbf{H}$. The condition for such a quadric to be inpolar to $\dot{y}\mu y=0$ is that the trace of the product matrix $\mu\mathbf{H}$, or, what is the same thing, that of $\mathbf{H}\mu$, should vanish. This trace is found by taking any row of μ and the *corresponding* column of $\mathbf{H}$, forming their vector product, and adding the six products so arising; a reference to 8.1 and 11.1 shows that this sum does in fact vanish identically.

12. Among the ∞^5 quadrics which contain F we have noticed, in § 10, ∞^4 line-cones, every chord of F being the vertex of such a cone. We also noticed, in § 7, ∞^2 plane-cones, namely, those cones which project the surface F from its tangent planes; the tangent primes of such a cone are those tangent primes of F which pass through that one of its tangent planes which is the vertex of the cone.

So, dually, among the ∞^5 quadric envelopes that are inscribed in F there are ∞^2 whose discriminants have rank 3 instead of their full rank 6; the primes which belong to such a quadric envelope consist of those which touch one of the conics on F. Any quadric which is outpolar to F must be outpolar to all these degenerate quadric envelopes; thus, in accordance with the usual interpretation of the vanishing of invariants of two quadrics when the discriminant of one of them falls below its full rank, *any secant plane π of F meets any outpolar quadric in a conic which is outpolar to the conic of F which lies in π.*

This being so, let T_1, T_2 be two points of F that are conjugate with respect to an outpolar quadric Ψ; the secant plane π which contains the conic γ of F passing through T_1 and T_2 meets Ψ in a conic ψ outpolar to γ. There must then be a triangle inscribed in ψ which is self-polar for γ and has T_1T_2 for one of its sides, so that the tangents of γ at T_1 and T_2 intersect on ψ. Whence

the tangent planes of F at any two points which are conjugate with respect to an outpolar quadric meet on this quadric.

The intersection of the two tangent planes of F also lies on C, and the argument is easily reversed to show that the two tangent planes of F which pass through any point common to C and an outpolar quadric have their points of contact conjugate with respect to this quadric.

13. Consider now those secant planes of F which meet an outpolar quadric Ψ in pairs of lines; it follows from § 5 that they are obtained by bordering μ by three rows and columns. They can also be identified by geometrical reasoning, and the combination of the two results furnishes an interesting form for a well-known contravariant of a ternary quartic.

Let a secant plane π of F meet Ψ in a pair of lines; π, like all other secant planes of F, meets Δ, the curve common to F and Ψ, in four points P_1, P_2, P_3, P_4 which correspond to four collinear points of δ. The points of the conic in which F is met by π are in (1, 1) correspondence with the points of the line on which the four points of δ lie. Now, since Δ lies on Ψ, the two lines in which π meets Ψ must join the four points of Δ in pairs; let the joins be P_1P_2 and P_3P_4. Since P_1P_2 lies on Ψ, P_1 and P_2 are two points of F that are conjugate with respect to Ψ; the tangent planes of F at P_1 and P_2 therefore, by § 12, meet on Ψ. But this point of meeting lies in π, being the intersection of the tangents to the conic of F at P_1 and P_2, and the only points of Ψ which lie in π and not on P_1P_2 lie on P_3P_4; hence the tangents to the conic at P_1 and P_2 must meet on P_3P_4. Similarly the tangents at P_3 and P_4 meet on P_1P_2. Wherefore P_1P_2 and P_3P_4 are conjugate chords of the conic, and the four points in which the conic meets Δ form, on the conic, a harmonic range.

Thus, *those secant planes which meet Ψ in pairs of lines correspond to those lines of σ which are cut in harmonic ranges by the quartic curve $\phi=0$.* It was seen in § 5 that the envelope of the lines in σ which correspond to secant planes of F that are touched by any quadric is of the sixth class; when the quadric is outpolar to F this envelope is the harmonic envelope of a quartic curve.

It follows that when μ is bordered by the rows U and the columns U′ the determinant of the resulting matrix gives, when equated to zero, the harmonic contravariant of ϕ.

Let us give two simple instances of this, by way of illustration.

If

$$\phi(x_1, x_2, x_3) \equiv x_1^4 + x_2^4 + x_3^4,$$

δ is the curve studied by Dyck and admitting a group of 96 linear self-transformations. The equation of its harmonic envelope is

$$\begin{vmatrix} 1 & . & . & . & . & . & u_1 & . & . \\ . & 1 & . & . & . & . & . & u_2 & . \\ . & . & 1 & . & . & . & . & . & u_3 \\ . & . & . & . & . & . & . & \tau^{-1}u_3 & \tau^{-1}u_2 \\ . & . & . & . & . & . & \tau^{-1}u_3 & . & \tau^{-1}u_1 \\ . & . & . & . & . & . & \tau^{-1}u_2 & \tau^{-1}u_1 & . \\ u_1 & . & . & . & \tau^{-1}u_3 & \tau^{-1}u_2 & . & . & . \\ . & u_2 & . & \tau^{-1}u_3 & . & \tau^{-1}u_1 & . & . & . \\ . & . & u_3 & \tau^{-1}u_2 & \tau^{-1}u_1 & . & . & . & . \end{vmatrix} = 0,$$

which gives clearly

$$-\begin{vmatrix} 1 & . & . \\ . & 1 & . \\ . & . & 1 \end{vmatrix} \begin{vmatrix} . & \tau^{-1}u_3 & \tau^{-1}u_2 \\ \tau^{-1}u_3 & . & \tau^{-1}u_1 \\ \tau^{-1}u_2 & \tau^{-1}u_1 & . \end{vmatrix} \begin{vmatrix} . & \tau^{-1}u_3 & \tau^{-1}u_2 \\ \tau^{-1}u_3 & . & \tau^{-1}u_1 \\ \tau^{-1}u_2 & \tau^{-1}u_1 & . \end{vmatrix} = 0,$$

or

$$u_1^2u_2^2u_3^2 = 0.$$

Thus the harmonic envelope of Dyck's curve consists of the three vertices of the triangle of reference, each counted twice. These points can of course be identified geometrically, independently of any co-ordinate system to which the curve may be referred; the curve has twelve points at which its tangents have four-point contact, and these twelve tangents are concurrent, in three groups of four, at the points in question.

Secondly, take

$$\phi(x_1, x_2, x_3) \equiv 4(x_2^3x_3 + x_3^3x_1 + x_1^3x_2),$$

so that δ is the curve studied by Klein and admitting a group of 168 linear self-transformations. The equation of its harmonic envelope is

$$\begin{vmatrix} . & . & . & . & . & \tau & u_1 & . & . \\ . & . & . & \tau & . & . & . & u_2 & . \\ . & . & . & . & \tau & . & . & . & u_3 \\ . & \tau & . & . & . & . & . & \tau^{-1}u_3 & \tau^{-1}u_2 \\ . & . & \tau & . & . & . & \tau^{-1}u_3 & . & \tau^{-1}u_1 \\ \tau & . & . & . & . & . & \tau^{-1}u_2 & \tau^{-1}u_1 & . \\ u_1 & . & . & . & \tau^{-1}u_3 & \tau^{-1}u_2 & . & . & . \\ . & u_2 & . & \tau^{-1}u_3 & . & \tau^{-1}u_1 & . & . & . \\ . & . & u_3 & \tau^{-1}u_2 & \tau^{-1}u_1 & . & . & . & . \end{vmatrix} = 0.$$

On expansion the known form

$$5u_1^2u_2^2u_3^2 - u_2^5u_3 - u_3^5u_1 - u_1^5u_2 = 0$$

for the harmonic envelope is found.

14. The determinant $|\mu|$ is well known as an invariant of a ternary quartic; when μ is bordered by the single row $u^{[2]}$ and the transposed column the ensuing determinant is a contravariant which was found by Clebsch in 1861. That this bordering of μ gives Clebsch's contravariant was observed, for the dual case of a ternary quartic in the line co-ordinates, by Scherrer in 1882; that the bordering by U gives the harmonic contravariant does not seem to have been perceived before.

It seems fitting to give the simple direct proof that these functions so formed are actually invariantly related to the quartic curve; the proof is immediate once the theorem of § 15 has been established and the equation 16.2 written down. The method of proof clearly admits of immediate extension to forms of any even order in any number of variables.

15. Suppose that the point co-ordinates are subjected to the linear transformation

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ m_{31} & m_{32} & m_{33} \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix},$$

or

$$x = m\xi. \tag{15.1}$$

This induces the transformation

$$x^{[2]} = m^{[2]}\xi^{[2]}, \tag{15.2}$$

the second induced matrix $m^{[2]}$ of m being defined by 15.2.

The transformation of differential operators consequent upon 15.1 is

$$\mathrm{D}_\xi = \left\{\frac{\partial}{\partial\xi_1}, \quad \frac{\partial}{\partial\xi_2}, \quad \frac{\partial}{\partial\xi_3}\right\} = \acute{m}\left\{\frac{\partial}{\partial x_1}, \quad \frac{\partial}{\partial x_2}, \quad \frac{\partial}{\partial x_3}\right\} = \acute{m}\mathrm{D}_x, \tag{15.3}$$

whence it follows that

$$\mathrm{D}_\xi^{[2]} = \left\{\frac{\partial^2}{\partial\xi_1^2}, \quad \frac{\partial^2}{\partial\xi_2^2}, \quad \frac{\partial^2}{\partial\xi_3^2}, \quad \tau\frac{\partial^2}{\partial\xi_2\partial\xi_3}, \quad \tau\frac{\partial^2}{\partial\xi_3\partial\xi_1}, \quad \tau\frac{\partial^2}{\partial\xi_1\partial\xi_2}\right\} = \acute{m}^{[2]}\mathrm{D}_x^{[2]}. \tag{15.4}$$

Now we have the equations

$$\tfrac{1}{12}\mathrm{D}_x^{[2]}\phi = \mu x^{[2]}, \qquad \tfrac{1}{12}\mathrm{D}_\xi^{[2]}\phi = \nu\xi^{[2]},$$

where ν is the matrix formed from the coefficients, of the form into which $\phi(x_1, x_2, x_3)$ is turned by the transformation 15.1, in exactly the same way as μ is formed from the coefficients of ϕ before it is subjected to the transformation. But the first of these equations is, by 15.2,

$$\tfrac{1}{12}\mathrm{D}_x^{[2]}\phi = \mu m^{[2]}\xi^{[2]}; \tag{15.5}$$

while the second is, by 15.4,

$$\tfrac{1}{12}\acute{m}^{[2]}\mathrm{D}_x^{[2]}\phi = \nu\xi^{[2]}. \tag{15.6}$$

Comparison of 15.5 and 15.6 shows that

$$\nu = \acute{m}^{[2]} \mu m^{[2]}, \tag{15.7}$$

and so the following theorem has been established:

If μ is a matrix of the form shown in 11.1, *and m any matrix of the third order, then $\acute{m}^{[2]}\mu m^{[2]}$ is of the same form as μ, where $\acute{m}^{[2]}$ is the transposed matrix of $m^{[2]}$, the second induced matrix of m.*

The work of § 16 would have no validity had this theorem not been established.

The proof of the theorem may be carried over almost *verbatim* to the case of the p^{th} induced matrix of m; if μ for the moment denotes the matrix of the coefficients in the $\frac{1}{2}(p+1)(p+2)$ p^{th} polars of a ternary form of even degree $2p$, then $\acute{m}^{[p]}\mu m^{[p]}$ is of the same form as μ. And the corresponding result holds for forms of even order in any number of variables, the order of m being equal to this number of variables. For quadratic forms the theorem becomes the trivial one that, if μ is symmetric, so is $\acute{m}\mu m$, where m is any matrix of the same order as μ.

16. Suppose now that m is non-singular, so that, when the point co-ordinates are subjected to the transformation 15.1, the line co-ordinates are subjected to the contragredient transformation

$$\begin{aligned}(u_1, \quad u_2, \quad u_3) &= (\omega_1, \quad \omega_2, \quad \omega_3)m^{-1}, \\ u &= \omega m^{-1}.\end{aligned} \tag{16.1}$$

It is a known property of the p^{th} induced matrix of a matrix m of order n that its determinant is (the result being attributed by Muir to Schläfli, 1851) the determinant $|\, m \,|$ raised to the power $\dfrac{(p+n-1)!}{(p-1)!\,n!}$. Hence, equating the determinants of the two sides of 15.7,

$$|\, \nu \,| = |\, m^{[2]} \,|\, |\, \mu \,|\, |\, m^{[2]} \,| = |\, m \,|^4 \,|\, \mu \,|\, |\, m \,|^4 = |\, \mu \,|\, |\, m \,|^8,$$

showing that $|\, \mu \,|$ is an invariant, of weight 8, of the ternary quartic ϕ.

Consider next Clebsch's contravariant

$$\begin{vmatrix} \mu & \acute{u}^{[2]} \\ u^{[2]} & . \end{vmatrix}.$$

Since, by 16.1, $\omega = um$, $\omega^{[2]} = u^{[2]}m^{[2]}$. Hence

$$\begin{pmatrix} \nu & \acute{\omega}^{[2]} \\ \omega^{[2]} & . \end{pmatrix} = \begin{pmatrix} \acute{m}^{[2]}\mu m^{[2]} & \acute{m}^{[2]}\acute{u}^{[2]} \\ u^{[2]}m^{[2]} & . \end{pmatrix}$$

$$= \begin{pmatrix} \acute{m}^{[2]} & . \\ . & 1 \end{pmatrix} \begin{pmatrix} \mu & \acute{u}^{[2]} \\ u^{[2]} & . \end{pmatrix} \begin{pmatrix} m^{[2]} & . \\ . & 1 \end{pmatrix},$$

the matrices being partitioned conformably, so that 1 denotes a single unit. Taking determinants gives

$$\begin{vmatrix} \nu & \omega^{[2]} \\ \omega^{[2]} & . \end{vmatrix} = |\, m \,|^4 \begin{vmatrix} \mu & u^{[2]} \\ u^{[2]} & . \end{vmatrix} |\, m \,|^4,$$

thus directly establishing the contravariance and displaying the weight 8.

Now let us take the form found for the harmonic envelope of ϕ, namely

$$\begin{vmatrix} \mu & \mathrm{U}' \\ \mathrm{U} & . \end{vmatrix} = 0,$$

and establish its contravariance directly. This is easily achieved if it is observed that

$$(u_1, u_2, u_3) \begin{pmatrix} u_1 & . & . & . & \tau^{-1}u_3 & \tau^{-1}u_2 \\ . & u_2 & . & \tau^{-1}u_3 & . & \tau^{-1}u_1 \\ . & . & u_3 & \tau^{-1}u_2 & \tau^{-1}u_1 & . \end{pmatrix}$$

$$= (u_1^2, \quad u_2^2, \quad u_3^2, \quad \tau u_2 u_3, \quad \tau u_3 u_1, \quad \tau u_1 u_2),$$

or

$$u\mathrm{U} = u^{[2]}, \tag{16.2}$$

and so also, if Ω denotes the matrix formed from ω in the same way that U is formed from u,

$$\omega\Omega = \omega^{[2]}. \tag{16.3}$$

But this last equation is

$$um\Omega = u^{[2]}m^{[2]},$$

and so, substituting for $u^{[2]}$ from 16.2,

$$m\Omega = \mathrm{U}m^{[2]}, \quad \text{giving} \quad \Omega = m^{-1}\mathrm{U}m^{[2]}.$$

It is now seen immediately that

$$\begin{pmatrix} \nu & \Omega' \\ \Omega & . \end{pmatrix} = \begin{pmatrix} m^{[2]}\mu m^{[2]} & m^{[2]}\mathrm{U}'m^{-1} \\ m^{-1}\mathrm{U}m^{[2]} & \vdots \end{pmatrix}$$

$$= \begin{pmatrix} m^{[2]} & . \\ . & m^{-1} \end{pmatrix} \begin{pmatrix} \mu & \mathrm{U}' \\ \mathrm{U} & . \end{pmatrix} \begin{pmatrix} m^{[2]} & . \\ . & m^{-1} \end{pmatrix},$$

and so, taking determinants,

$$\begin{vmatrix} \nu & \Omega' \\ \Omega & . \end{vmatrix} = |\, m \,|^{4-1} \begin{vmatrix} \mu & \mathrm{U}' \\ \mathrm{U} & . \end{vmatrix} |\, m \,|^{4-1} = |\, m \,|^6 \begin{vmatrix} \mu & \mathrm{U}' \\ \mathrm{U} & . \end{vmatrix}.$$

This establishes the contravariance, and is in accordance with the known fact that the harmonic envelope is of weight 6, its Aronhold symbol being $(bcu)^2(cau)^2(abu)^2$.

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XIII.—Some Disputed Questions in the Philosophy of the Physical Sciences.

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MORE than two thousand years ago the Greek philosophers raised certain questions, which are still undecided, about the origin and character of knowledge regarding the external world. After a period of comparative quiet, the discussion has become very active recently, under the stimulus of the new discoveries in mathematical physics; and, in particular, a lively debate is in progress at the present moment between Sir Arthur Eddington and Dr Harold Jeffreys of Cambridge, Professor Milne of Oxford, Sir James Jeans, and Professor Dingle of the Imperial College, the subject being the respective shares of reason and observation in the discovery of the laws of nature. I propose this afternoon to offer some remarks on the history and present state of this controversy.

It is admitted by everyone that the mathematical pre-calculation of natural events is conceivable only because the world is rationally made: and that the ideal of science is to discover laws and equations sufficient to make every physical event predictable, so that mankind will some day possess a corpus of purely mathematical relations, capable of representing and foretelling every happening in the inanimate external world. Now it is characteristic of any branch of mathematics, that the whole of it can be worked out from a few definitions and assumptions set down at the beginning. The question therefore arises, what are the fundamental data or postulates from which the complete set of laws of the material universe can be deduced by pure mathematics? And are these data or postulates furnished to us by the senses—by observation and experiment—or are they self-evident truths, revealed and assured to us by intuition?

This fundamental problem of the philosophy of nature was first conceived and discussed by the ancient Greeks, whose judgment on it can be very clearly seen in their treatment of geometry. Geometry is the science of spatial relations in the external world and so is essentially a part of natural philosophy: and the knowledge of many geometrical properties

was, doubtless, originally derived from observation: thus according to one conjecture, Thales became convinced that the angle in a semicircle is always a right angle, by gazing at tiles which showed, as part of the ornamental design, a rectangle inscribed in a circle. This earliest stage of geometry corresponded to that now generally imposed on school children, who are taught to draw figures with ruler and compasses, and to measure angles of triangles with protractors, before being introduced to the deductive treatment of the subject. When the logical connexion of different theorems was established by Pythagoras and his school, the idea gained ground that it might be possible to link up the entire science of geometry into a chain of propositions obtained by syllogistic reasoning from a small number of original premisses. The great question was, from whence were these premisses derived? Plato believed that they could all be obtained by pure intellection; and Euclid drew up a list of five "common notions" or axioms and five "postulates," from which he professed to demonstrate all the results of geometry as logical conclusions. The "common notions," such as "Things that are equal to the same thing are equal to each other," were presented as self-evident truths which belong alike to every science. The postulates, such as "Let it be granted that a circle may be described with any centre and diameter," were put forward as admitted feasibilities, peculiar to the science of geometry. Neither for common notions nor for postulates was any proof offered: the disciple was expected to know by intuition that they were necessary, that things could not be otherwise. Thus the Greek philosophers taught that although geometry was a science relating to the sensible external universe, it could be built up completely without having recourse at any stage to quantitative observation.

Even in the ancient world there was some uneasiness about the validity of this doctrine. One of Euclid's premisses was the famous "parallel-postulate," "If a straight line, falling on two straight lines, makes the interior angles on the same side less than two right angles, the two straight lines, if produced indefinitely, meet on that side on which are the angles less than two right angles." This is not a particularly simple statement—are we justified in accepting it without proof as true? Some of the Greeks, and many later mathematicians down to the nineteenth century, were inclined to suspect that, though true, it was not a primary truth which could be referred immediately to the intuition, but a theorem which could be established by deductive reasoning from simpler self-evident assumptions. Accordingly they tried to discover some more plausible axiom from which it could be derived: an endeavour which met with a certain degree of success when in the seventeenth century Wallis

showed that if the existence of triangles different in size but similar in shape could be assumed, then the parallel-postulate could be proved and Euclidean geometry re-constituted. Even this, however, was not altogether satisfactory: for it cannot be claimed that a belief in similar triangles is a necessity of human thought. This may easily be shown as follows: take a globe such as the earth, and mark any number of points on it, such as London, Edinburgh, Stockholm, and so forth; then mark on the globe the tracks of airmen flying directly between these points. The three tracks joining any three points to each other form what is called a "spherical triangle," and the angles which the tracks make with each other at the three points are called the "angles of the spherical triangle"—evidently the angles determine the *shape* of the triangle. Now consider a particular spherical triangle formed on the earth in the following way: one vertex N is to be at the north pole, and the other two vertices A and B are to be on the equator. The sides NA and NB will then be meridians of longitude on the earth, and the side AB will be part of the equator. It is evident that the angles at A and B are right angles: and hence the sum of the angles of the triangle is equal to two right angles together with the angle at N. Now, since A and B are on the equator, and N is the pole, the area of the spherical triangle is evidently proportional to the angle at N: and thus we have for triangles of this kind the result that the sum of the three angles is greater than two right angles, by an amount which is proportional to the area of the triangle. This result may be shown to be true for any spherical triangle whatever—which proves that two spherical triangles of different sizes cannot possibly have the same angles, since the angle-sum must be greater for the larger than for the smaller triangle.

Thus there are no similar triangles in spherical geometry—that is to say, the mind can conceive of a type of geometry in which similar triangles do not exist. Thus Wallis's postulate is not something which we are compelled to believe by reason of the structure of our minds.

We are therefore forced to the conclusion that Euclidean geometry cannot be deduced from self-evident axioms; and hence the question as to whether it is the true geometry of the actual world can only be settled by making measurements and examining how far they agree with the theoretical predictions. Geometry is a branch of experimental knowledge. This fact, so important from the point of view of the philosophy of the physical sciences, obtained general acceptance very slowly—the belief that the Euclidean system is necessarily true was unquestioned for two millenniums, and became finally discredited only in the second half of the nineteenth century.

We have now to inquire whether the ancient Greeks thought it possible that the other sciences might, like geometry, be deduced logically from premisses self-evident to the intuition. On this point Aristotle, at any rate, was quite clear. "The principles which lie at the basis of any particular science," he says,* "are derived from experience (ἐμπειρία): thus it is from astronomical observation that we derive the principles of astronomical science." He poured scorn on what he called ἀπειρία †—that is, the state of those who devote themselves entirely to abstract reasoning from intuitive postulates, and are indifferent to facts. And he insisted on the vital importance of investigating any subject scientifically (φυσικῶς)—that is to say, by the study of sensible objects in nature ‡—rather than dialectically (λογικῶς), that is, by pure deduction from unproved assumptions. The primary elements of true science must be the result of induction (ἐπαγωγή) based on phenomena, proceeding from particulars to universals. Sense-perception is indispensable to induction, and induction to discovery of the laws of nature.

The opinions of Aristotle on this subject were accepted by the schoolmen who developed his philosophy in the Middle Ages. St Thomas Aquinas himself said § that the only secure foundation is experiment, and many of the scholastics made accurate observations and rational deductions from them. It may seem at first sight surprising that under these circumstances the thirteenth and fourteenth centuries did not anticipate the brilliant scientific discoveries of the seventeenth and eighteenth. The chief reasons were, I think, these. In the first place, the outstanding example of Euclidean geometry, at that time the most fully developed branch of science, whose results were believed to have been obtained without any dependence on observation or experiment, encouraged among many of the schoolmen a tendency to neglect the principles so clearly set forth by Aristotle and St Thomas, and to concentrate their interest on pure deduction; and this was not uncongenial to their general outlook, for, recognising in the external physical order an expression of the Divine Reason, they inferred that the human reason was the proper organ for its investigation. In the second place, it is to be remembered that Aristotle was a biologist rather than a physicist; and the biologist, at least in the earlier stages of biology, classifies rather than measures. The emphasis in Aristotelianism was decidedly on classification; and the idea of mathematical precision in quantitative measurements, which is

* *Prior Analytics*, I, 30.

† *De Gen. et Corr.*, I, 2.

‡ *Ibid.*

§ *Cf.* St Thomas, *Commentary on Aristotle's "Physics,"* Lib. VIII, Cap. 1, Lect. 3, 4.

the fundamental principle of modern physics, scarcely entered into the conception of Induction which the Scholastics inherited from the Peripatetics. Again, the primitive biologist depends on observation (the study of those appearances which nature presents of her own accord) rather than on experiment (the artificial production of effects under conditions specially selected and controlled by the experimenter). Simple observation will carry the zoologist a long way: but it is insufficient for the purposes, say, of the chemist, who has to disentangle all the antecedents and concomitants of the reaction he has observed. Hence Aristotle and his followers were much more at home in the biological than in the physical domain.

The great archetype of modern scientific discovery was the Newtonian law of gravitation. How wonderful an achievement it was may be seen by comparing it with the concept of nature as a system of vortices proposed only forty years earlier by Descartes, in which there is no attempt to obtain concordance between the deductions of the theory and experimental data. With Newton, on the other hand, we find the complete technique of research as it is practised to-day—namely, first the accurate measurement of phenomena; secondly, the imagination of a physical hypothesis likely to account for them, then the working out by mathematics of the detailed consequences of the hypothesis; and lastly, the comparison of the results of these calculations with the observations. When we reflect that by this investigation the entire cosmos was for the first time shown to conform to prediction based on a mathematical law, and that it provided the model and pattern for all subsequent progress, we cannot but confirm the judgment engraved below the statue in Trinity chapel: *Newton, qui genus humanum ingenio superavit.*

From the time of Newton until quite recently the principle that science rests fundamentally on observation and experiment has been unchallenged. In order to understand the present revolt against it, let us first consider some striking cases in which a mathematician, working in his study without any contact with laboratories, has been able to predict the existence of wholly new and unexpected phenomena in the external world.

A typical example is Hamilton's discovery of conical refraction. If we mark a dot on a piece of paper, and look at it through a crystal of Iceland spar, we see in general not one dot but two: this is because the spar has the property called *double refraction*. In 1821 the French physicist Fresnel discovered the equation of the *wave-surface*, or locus at any instant of a disturbance generated at a particular point at some previous instant, in a doubly refracting crystal; but he did not study

the geometrical features of the surface as a mathematician would do. This was precisely what Hamilton did. He found that Fresnel's surface had some remarkable singularities, such as sharp peaks like the vertex of a cone, at each of which it had an infinite number of tangent-planes; and from the existence of these mathematical peculiarities in the wave-surface he inferred the existence of a corresponding optical phenomenon of a most amazing kind, namely, that a ray of light within the crystal would, under certain circumstances, be divided on emergence into an infinite number of rays, constituting a conical surface, while a single ray in air, incident on the crystal, might give a cone of rays inside the crystal. It should therefore be possible to arrange an experiment so that a dot on a sheet of paper, when viewed through the Iceland spar, should appear not as two dots, but as a complete circle. This prediction was immediately verified.

Since the beginning of the twentieth century many novel and remarkable effects in physics have been discovered by mathematicians. Einstein's predictions of the bending of light-rays by the sun's attraction, and of the red-shift of spectral lines emitted in a strong gravitational field, are two notable instances. These were deduced from his general theory of relativity, which was entirely mathematical—that is to say, it did not originate in, or depend on, any new experiments or observations. Yet another striking discovery of the same type was the recognition that ordinary hydrogen gas is a mixture of two different kinds of molecules. When it is remembered that for a century and a half hydrogen has been familiar to every schoolboy beginning science, and has been the subject of innumerable experiments by highly trained investigators, the announcement of its composite character came as a great surprise. The mathematical reasoning involved turned on a distinction which is made in algebra: if we have two algebraic quantities x and y, we can form from them certain expressions such as $x+y$ whose value is unaltered when x and y are interchanged with each other; these are said to be *symmetric* in x and y. We can also form expressions such as $x-y$ whose value is reversed in sign when x and y are interchanged; these are said to be *skew* in x and y. Now it was found, in 1927, that the mathematical equations which represent the conditions of existence of the hydrogen molecule possess two different solutions, of which one is symmetric and the other is skew. It followed that there must be two different kinds of hydrogen molecule, to which the names *para-hydrogen* and *ortho-hydrogen* were given. These two tautomers behave in exactly the same way as regards the formation of chemical compounds, which explains why the chemists had never distinguished or separated them; but the specific heat

of para-hydrogen is greater at low temperatures than that of ortho-hydrogen; and their boiling-points and conductivities are also different. Hydrogen gas prepared by the usual chemical processes consists of one-quarter para-hydrogen and three-quarters ortho-hydrogen.

We have now to consider whether the doctrine that science must be based on observation needs any modification, in view of the fact that discoveries in physics of the most unexpected kind and of the greatest importance are frequently made by mathematicians who have never performed, or even seen, an experiment in their lives. Obviously the dominant principle to be taken into account here is that the world is *rational*, the different phenomena being interconnected logically so that if we have found by observation a certain number of them, we can deduce the others by pure reasoning without making any fresh observations. Thus if we find by experiment that the coefficient of magnetisation of a piece of soft iron decreases as its temperature increases, we can predict that a magnet will become heated when it is moved from weak to strong regions of the magnetic field.

We are thus led up to the most important problem of Natural Philosophy: What is the minimum set of observational data which is sufficient to form a basis for the whole edifice of physical theory? Certain eminent living physicists now give to this question the amazing answer: None at all! That is to say, they repudiate altogether the principle enunciated by Aristotle, St Thomas, and Newton, and return to the attitude which was maintained for so long with regard to geometry, and which they now propose to extend to the whole of physical science.

In order to appreciate their standpoint, so unexpected and paradoxical as it must at first seem, let us consider one particular branch of physics, namely, the subject of electricity and magnetism. The early experiments and deductions in frictional electricity by Gilbert, Gray, Du Fay, Watson, Franklin, and Cavendish were completed by the measurement of the attraction between two electrically charged bodies, and its representation by the law generally known by the name of Coulomb, but actually first discovered by Priestley. In magnetism the recognition of magnetic poles by Petrus Peregrinus in the thirteenth century led eventually to the determination by Michell of the law of force between them. In the first half of the nineteenth century Oersted found that an electric current generates a magnetic field, Ampère showed that a ponderomotive force is exerted between two circuits carrying electric currents, and Faraday discovered the induction of currents. These researches, spread over six centuries and pursued incessantly during the last two of them, were experimental, and each of the physical properties concerned rested

independently on its own observational basis. The various results were finally combined and consummated in the general electromagnetic theory of Maxwell, which comprehended all known electric and magnetic phenomena.

Before Maxwell, the story is one of performing experiments and devising formulæ to represent the results. But the post-Maxwellian period is wholly different in character. Maxwell's synthesis went beyond the experiments and predicted the existence of something that had never as yet been observed, namely, electromagnetic waves—what are now called wireless waves—and the correctness of this inference was verified after the death of Maxwell by the celebrated researches of Hertz. It may be said generally that experimental investigations in electricity from 1870 to 1900, such, for instance, as Rowland's work on fields produced by the motion of electric charges, were directly inspired by Maxwell's theory, and that the mathematical equations were capable of furnishing the results of the experiments beforehand.

The change in the method of discovery after Maxwell may be illustrated by a simple analogy. Suppose that a map of Scotland is pasted on stiff cardboard and then cut up into small irregular pieces, so that it can be used as a jigsaw puzzle. Anyone who tries to solve the puzzle does not at first know what is represented, and his only possibility of procedure is to find pieces which fit into each other and so constitute larger parts of the whole. After a time, however, he will have progressed sufficiently to be able to guess that what is represented is Scotland, and from that time onwards he completes the work not by finding pieces which fit into each other, but by using his *à priori* knowledge of Scotland to put every fragment into its proper place. These two methods may be likened to the two types of research in physical science: the earlier, proceeding step by step by experiment in special topics; and the later, knowing *à priori* what ought to be, because a guiding principle is now available for the whole, permitting the extension of knowledge by purely rational methods. When the work of fitting together the puzzle is completed, the picture alone remains as significant—that is to say, in physics, the mathematical theory.

The logical unity achieved in electrical science by Maxwell had this consequence, that the vast majority of the experiments, on which it had originally been built up, were now superfluous, and the number of ultimate independent observational facts necessary for its establishment was very small. Eventually they were reduced to only one, namely, that when a hollow metallic vessel is electrified there is no electric field in the air inside. From this single datum, combined with *à priori* principles such

as the axiom of relativity, it is possible to derive mathematically, first, the inverse-square law of force between electric charges at rest, then (by considering these charges in motion relative to an observer) to deduce the existence of magnetic force, and finally to obtain in Maxwell's form the general equations of the electromagnetic field.

The single experimental fact on which electric and magnetic science can be founded may be put in the form, "It is impossible to set up an electric field in any region of space by enclosing the space in a hollow conductor of any shape or size and charging the outside of the conductor"; and in this form it bears a certain family likeness to other statements on which important branches of physics have been based, such as the postulate of thermodynamics (from which a great part of physical chemistry is derived), "It is impossible to derive mechanical effect from any portion of matter by cooling it below the temperature of the coldest of the surrounding objects"; or the postulate of Relativity, "It is impossible to detect a uniform translatory motion, which is possessed by a system as a whole, by observations of phenomena taking place wholly within the system"; or the postulate (which plays an important part in the explanation of homopolar bonds in chemistry) that "It is impossible at any instant to assert that a particular electron is identical with some particular electron which had been observed at an earlier instant"; or the postulate of Imperfect Definition in quantum mechanics, "It is impossible to measure precisely the momentum of a particle at the same time as a precise measurement of its position is made." Each of these statements, which I propose to call *Postulates of Impotence*, asserts the impossibility of achieving something, even though there may be an infinite number of ways of trying to achieve it. A postulate of impotence is not the direct result of an experiment, or of any finite number of experiments; it does not mention any measurement, or any numerical relation or analytical equation; it is the assertion of a conviction of the mind, that all attempts to do a certain thing, however made, are bound to fail. We must therefore distinguish a postulate of impotence, on the one hand, from an experimental fact; and we must also distinguish it, on the other hand, from the statements of Pure Mathematics, which do not depend in any way on experience, but are necessitated by the structure of the human mind; such a statement, for instance, as "It is impossible to find any power of two which is divisible by three." We cannot conceive any universe in which this statement would be untrue, whereas we can quite readily imagine a universe in which any physical postulate of impotence would be untrue.

It seems possible that while physics must continue to progress by

building on experiments, any branch of it which is in a highly developed state may be exhibited as a set of logical deductions from postulates of impotence, as has already happened to thermodynamics. We may therefore conjecturally look forward to a time in the future when a treatise on any branch of physics could, if so desired, be written in the same style as Euclid's *Elements of Geometry*, beginning with some *à priori* principles, namely, postulates of impotence, and then deriving everything else from them by syllogistic reasoning.

With our minds thus prepared, let us approach the doctrines that Sir Arthur Eddington and Professor E. A. Milne have lately given to the world.

Milne begins by regarding the universe as an aggregate, of which one member is our galactic system of stars, the other members being the extra-galactic nebulæ, each of which is actually a vast system of stars. He represents this aggregate in an idealised fashion as a collection of particles, each particle corresponding to one nebula; and each particle is supposed to carry an observer, equipped with a clock and a theodolite and with apparatus for sending and receiving light-signals. Milne then postulates that this ideal universe satisfies what he calls the *cosmological principle*, namely, that the map of the world which any particle-observer constructs from his own observations is identical with the map which any other particle-observer constructs from *his* own observations—that is, each sees the same sequence of views of the life-history of the universe.

The cosmological principle evidently bears a fairly close resemblance to the axiom of relativity, and may, like it, be expressed in the form of a postulate of impotence. Thus what Milne does essentially is to assume a postulate of impotence and then deduce its logical consequences, much as the science of thermodynamics is deduced from the Second Law; or, to go further back, much in the same way as Euclid deduced the whole of geometry from his common notions and postulates. Indeed, Milne describes his work as an attempt to do for systems of particles in motion what the Greeks did for static assemblages of points. He arrives at a detailed account of the life-history of the universe, and derives the laws of dynamics, the inverse-square law of gravitation, and the equations of electromagnetism.

The work of Sir Arthur Eddington is, on the philosophical side, more extensive and much more difficult to characterise than that of Professor Milne. In two books, published in 1936 and 1939 respectively, he has set forth systematically what is offered as a complete philosophy of physical science.

The origin of his distinctive ideas may, I think, be traced to the

circumstance that in the upbuilding both of relativity and of quantum mechanics the question, "What is it that we really observe?" has played a great part. The critical examination arising out of this question showed that the older physicists had been accustomed to talk about things that are inaccessible to observation, whose existence and meaning are therefore highly doubtful. It came to be recognised, for instance, that no significance can be attached to the term "absolute velocity in space," and that it is impossible to imagine an experiment which could follow continuously the motion of an electron in an atom. These advances towards precision of thought, and the remarkable consequences which followed from them, made on Eddington a profound impression, and his aim in recent years has been to extend them, by making a thoroughgoing study of the nature and limitations of our knowledge of the physical sciences.

His attitude may be indicated by an analogy. Suppose a sea-fisherman invariably finds on examining his catch that no fish in it is less than two inches long. He might be inclined to assert as a fundamental law of zoology that all fishes are at least two inches in length, did he not know the real reason for the observed fact, namely, that he was fishing with a net of two-inch mesh. In interpreting the analogy, we may compare the net to the methods of research employed in physics, and the two-inch minimum for fishes to the laws physicists have discovered, which have in the past been regarded as describing properties of the real world, but which may be merely the necessary consequences of the procedure used in obtaining them. Hence Eddington's insistence on the importance of a proper investigation of the sensory equipment with which we observe, and the mental equipment by aid of which we formulate the results of observation. From this scrutiny he derives certain conclusions which he calls *epistemological principles*—principles relating to the method or ground of knowledge—and which, he asserts, are adequate to supersede entirely all observation and experiment as the basis of physical science.

This claim is so astounding that it must be stated in his own words. "I believe," he says,* "that the whole system of fundamental hypotheses [of physics] can be replaced by epistemological principles. Or to put it equivalently, all the laws of nature that are usually classed as fundamental can be foreseen wholly from epistemological considerations." "This means," he adds, † "that the fundamental laws and constants of physics are wholly subjective, being the mark of the observer's sensory and intellectual equipment on the knowledge obtained through such equip-

* *The Philosophy of Physical Science*, 1939, p. 56.

† *Ibid.*, p. 104.

ment, for we could not have this kind of *à priori* knowledge of laws governing an objective universe." "An intelligence," he says in another place,* "unacquainted with our universe, but acquainted with the system of thought by which the human mind interprets to itself the content of its sensory experience, should be able to attain all the knowledge of physics that we have attained by experiment. He would not deduce the particular events and objects of our experience, but he would deduce the generalisations we have based on them. For example, he would infer the existence and properties of sodium, but not the dimensions of the earth."

The doctrine thus proclaimed is so entirely contrary to all the received ideas of the nature of science, that it will certainly not be accepted without a careful examination. In the first place, we may remark that Eddington's epistemological principles cannot be absolutely antecedent to all experience of the external world, for without some such experience it would be impossible to attach any meaning to the language employed (this indeed he admits). In the second place, we may observe that the notion of an "intuitive truth" has a sense that is relative rather than absolute. Take, for example, the statement that the sum of 1 and 5 is equal to the sum of 2 and 4. We are accustomed to speak of this as a proposition necessitated by pure logic, and indeed, to those who have mastered Whitehead and Russell's *Principia Mathematica*, so it is; but it is not an intuitive truth to a savage who cannot count beyond four, and there is no doubt that it came to be accepted only as the result of a long development of culture, in which the operation was frequently performed of adjoining one object to five objects and observing that the same collection could be obtained by adjoining two objects to four objects. That is to say, a statement which we believe from inescapable necessity was originally a conviction arrived at from countless experiments. We must therefore envisage the possibility that the postulates of impotence and the Eddingtonian principles, to which most physicists, if they would assent to them at all, would assent on the ground of observation, may, by a more highly cultured or more penetrating mind, be perceived as self-luminous and irresistible truths, for which all proof would be superfluous. In Indian philosophy, I understand, there is a doctrine which asserts the possibility of an ineffable union with the infinite intelligence; but one hesitates to put forward, as the solution of the difficulties attending the epistemological assumptions, the suggestion that some such condition is already enjoyed by their distinguished author.

In order to examine more closely the validity of the thesis before us, let us consider in detail one of its principles, and the physical results

* *Relativity Theory of Protons and Electrons*, 1936, p. 327.

which Eddington claims to have deduced from it. An assumption which he specially mentions* is the "special relativity principle." This, and some of his other epistemological axioms, happen to be closely connected with some of what I called "postulates of impotence"; but the concept of a postulate of impotence is quite different from Eddington's concept of an epistemological principle, since the postulates of impotence are generalistions from experiment, which epistemological principles expressly are not; and also because some of the latter are not assertions of inability to do anything.

Take, then, the special relativity principle, and a physical result which Eddington offers as a deduction from it, namely, the law of increase of mass with velocity. The story of this law is a curious one. As everybody knows, in Newtonian dynamics every particle of matter has a definite *mass* or measure of inertia, which is invariable—it does not change with the situation, temperature, or velocity of the particle. It was, however, shown long ago by J. J. Thomson and Lorentz, from electromagnetic theory, that when a body is electrified its inertia when in motion must be greater than when it is uncharged, and, moreover, that the additional mass thus created must increase when the body's velocity increases. An expression for the increase of the mass of an electron with its velocity was found mathematically by Lorentz, and was confirmed in the laboratory by experiments on the deflection of high-speed electrons in an electric and magnetic field. In the theoretical derivation, it was assumed that the mass of the electron was entirely electromagnetic, and the observational verification of the formula was naturally taken to be a confirmation of this assumption. It was therefore with amazement that physicists learnt, soon after the discovery of the theory of relativity, that Lorentz's formula had really nothing to do with the electromagnetic theory by which it had been obtained, and that it was true not merely for electrons, but for all bodies whether electrified or not—it is in fact simply an immediate consequence of relativist dynamics.

Now let us hear Eddington on the subject. "As an example," he says,† "we may take the law of increase of mass with velocity, which has been the subject of many famous experiments. It is now realised that this law automatically results from the engrained form of thought which separates the fourfold order of events into a threefold order of space and an order of time. When knowledge is formulated in a frame which compels us to separate a time dimension from the fourfold order to which it belongs, a component called the mass is correspondingly separated

* *The Philosophy of Physical Science*, p. 39.

† *Ibid.*, p. 116.

from the fourfold vector to which it belongs: and it requires no very profound study of the conditions of separation to see how the separated component is related to the rest of the vector which prescribes the velocity. It is this relation which is rediscovered when we determine experimentally the change of mass with velocity."

This account of the matter ignores everything in the deduction of the mass-velocity formula except one particular step, namely, the separation of the temporal component from a vector in space-time. But, before this separation can be effected, we must know all about the vector—that is, we must already be in possession of relativist dynamics. Now relativist dynamics is based in part on the Axiom of Relativity, which may perhaps be called an epistemological principle (though actually it was arrived at only as an inference from an enormous number of experiments); but the Axiom of Relativity taken alone is certainly not sufficient to yield all that is wanted. There is not, so far as I know, any treatment of relativity-theory which does not in some way, directly or indirectly, make use of the assumption that the formulæ of relativist dynamics reduce to the formulæ of Newtonian dynamics when the velocities concerned are small. Indeed, without this assumption it does not seem possible to set up the desired connexion between the mathematical equations, on the one hand, and phenomena in the external world, on the other. But Newtonian dynamics was undoubtedly founded on experiment, and it is difficult to see that Eddington has provided any other derivation.

The perplexity generated by his account of the laws of nature is not lessened by studying his work on the fundamental constants. "Not only the laws of nature," he says,* "but the constants of nature can be deduced from epistemological considerations, so that we can have *à priori* knowledge of them." He actually obtains four constants, namely, the number of particles in the universe, the ratio of the electric to the gravitational force between a proton and an electron, the ratio of the mass of the proton to the mass of the electron, and what is called the fine-structure constant, which is of great importance in atomic physics. The last three of these were already known experimentally, but the first, the number of particles in the universe, which he calls the *cosmical number*, is hardly susceptible of observational verification; it is derived by epistemological reasoning of the purest kind, being in fact equated to the number of independent quadruple wave-functions, which is $2 \times 136 \times 2^{256}$. Naturally the question arises as to what exactly is meant by the "number of particles in the universe." At the time when the theory was originally propounded it was believed that all atoms are composed of two kinds of elementary

* *The Philosophy of Physical Science*, p. 58.

particles, namely, protons and electrons; and Eddington defined the cosmical number to be simply the number of protons plus the number of electrons in the world. Since then the situation has become more complicated: a fresh elementary particle, the neutron, was found experimentally in 1932; another, the positron, theoretically in 1930 and experimentally in 1933; another, the neutrino, theoretically in 1931; and yet another, the meson, theoretically in 1935 and observationally in 1937. In the light of these discoveries, some re-interpretation of the cosmical number was evidently necessary. Eddington has decided * that a neutron should count as two particles, and a positron as minus one, so that the creation and annihilation of electrons and positrons in pairs will not affect the total. What is to be done about mesons and neutrinos is not yet settled. It must be said that readjustments of this kind are too suggestive of patching-up to be altogether satisfying. However, their author's confidence has in no way diminished, and he now claims † that the cosmical number determines the speed of recession of the distant nebulæ and the range of action of the forces between the particles in an atomic nucleus; and that all these things, being accounted for epistemologically, are *ipso facto* subjective—they are demolished as part of the objective world.‡

From the Greek nature-philosophers to Eddington the wheel has come full circle. Each of them believed, for example, that the value of the sum of the angles of a triangle is a necessary consequence of the constitution of the human mind. But they do not agree on what the value is. The Greeks said it was two right angles, while Eddington says that for triangles of astronomical size it is greater than two right angles by an amount depending on the size of the triangle.

A word may be said in conclusion on the question as to whether Eddington's general attitude may be subsumed under any of the historical philosophical schools. The two philosophers to whom he acknowledges himself indebted are Kant § and Bertrand Russell ‖—an oddly-assorted pair, since Russell has devoted much of his activity to confuting Kant. What Eddington admires in the philosopher of Königsberg is the *à priori* doctrine regarding space, to which his own principles bear a certain resemblance. This affiliation is not likely to commend the new epistemology to mathematicians, who still remember with bitterness the harm

* *The Philosophy of Physical Science*, p. 170.
† *Ibid.*, p. 177.
‡ *Ibid.*, p. 59.
§ *Ibid.*, p. 188.
‖ *Ibid.*, p. 151.

that was done in the early days of non-Euclidean geometry by Kantian opposition; and, perhaps partly on this account, Eddingtonian views have not hitherto made many converts. The admiration which is universally accorded to the genius of their author is assuredly not diminished by his stupendous attempt to base the study of the external world on a new foundation: but on the work itself the verdict as yet is *not proven*.

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XIV.—Further Investigations in Factor Estimation.

By **D. N. Lawley**, B.A., Moray House, University of Edinburgh. *Communicated by* Professor GODFREY H. THOMSON.

(MS. received October 21, 1941. Read December 1, 1941.)

1. IN a previous article (Lawley, 1940) the present writer applied the method of maximum likelihood to the problem of estimating the loadings of a set of tests in a number of factors. The assumption there made was that the observed test scores and also the factors involved were normally distributed over the population of persons tested. This assumption has been criticised by Gale Young (1941), who has pointed out that nothing need be implied concerning the distributions of the observed scores over a population of individuals, but that it is only necessary to make assumptions concerning the error distributions. It may, however, be remarked that in other branches of statistical inquiry it is a usual procedure to assume that the observed variates are normally distributed throughout the population, and that even when this assumption is not perfectly fulfilled the practical effects are generally not serious provided that the departures from normality are not too large. In fact, as we shall indicate, the method of estimation previously given does in practice lead to results which are in accord with those obtained by other methods commonly in use. It is of course true that the estimation of the factor loadings of the tests administered is only half the problem, and that the ultimate object is to estimate the amounts of the factors possessed by the various individuals tested. Nevertheless, once the factor loadings of the tests have been found, the individual factor measurements may then be estimated either from the equations given by Bartlett (1937) or from those of Thomson (1936).

In his solution of the factor problem Gale Young makes, for practical reasons, what is equivalent to the supposition that the error variance on a hypothetical infinity of trials is independent of the individual but depends only on the test. He assumes, however, that the error variances have been previously determined from other data. In the present paper we shall show that this last assumption is not strictly necessary since, given the scores of a group of individuals on a set of tests, it is in theory possible to estimate simultaneously not only the factor measurements

of the individuals and the factor loadings of the tests but also the error variances.

2. Let us suppose that n tests are given to a group of N individuals, and let us make the hypothesis that the scores obtained by them depend on the presence of m common factors. Then, if $x_{i\alpha}$ is the score of the α^{th} individual in the i^{th} test, we shall assume that

$$x_{i\alpha} = \sum_{r=1}^{m} (\lambda_{ri}\phi_{r\alpha}) + e_{i\alpha}, \tag{1}$$

where λ_{ri} is the loading of the i^{th} test and $\phi_{r\alpha}$ the measure of the α^{th} individual in the r^{th} factor.

It will be further assumed that the component $e_{i\alpha}$ is due to random error and that, for a given value of i, $e_{i\alpha}$ is distributed normally with a variance σ_i^2 which is independent of the value of α.* We may also, without loss of generality, suppose that $x_{i\alpha}$, $e_{i\alpha}$, and $\phi_{r\alpha}$ are measured from the corresponding sample means, so that

$$\sum_{\alpha=1}^{N} (x_{i\alpha}) = \sum_{\alpha=1}^{N} (e_{i\alpha}) = 0$$

for all values of i,

$$\sum_{\alpha=1}^{N} (\phi_{r\alpha}) = 0$$

for all values of r.

The joint distribution of the set of errors $\{e_{i\alpha}\}$ then takes the form

$$L \prod_{i,\alpha} (de_{i\alpha}),$$

where L is the likelihood function and is equal to

$$\frac{1}{(2\pi)^{\frac{1}{2}n(N-1)}(\sigma_1\sigma_2 \ldots \sigma_n)^{N-1}} e^{-\frac{1}{2}\sum_i \left\{\frac{1}{\sigma_i^2} \sum_\alpha (e_{i\alpha}^2)\right\}}.$$

Using equation (1) we find that

$$\log_e L = -(N-1)\sum_i \log_e \sigma_i - \tfrac{1}{2}\sum_i \left\{\frac{1}{\sigma_i^2}\sum_\alpha \left[x_{i\alpha} - \sum_r (\lambda_{ri}\phi_{r\alpha})\right]^2\right\} + \text{a constant.}$$

Efficient estimates of the unknown parameters $\{\sigma_i^2\}$, $\{\lambda_{ri}\}$, and $\{\phi_{r\alpha}\}$ may as usual be found by choosing values of these quantities which make L a maximum. In order to do this we differentiate $\log_e L$ with respect to each of the parameters in turn and equate the results to zero. Denoting the above estimates by $\{s_i^2\}$, $\{l_{ri}\}$, and $\{f_{r\alpha}\}$ respectively (so as

* We may, if desired, suppose $e_{i\alpha}$ to include a specific factor, provided that such specifics are assumed to be normally distributed over the population of individuals tested.

to make a distinction between population parameters and their sample estimates), we are thus led to the equations

$$s_i^2 = \frac{1}{(N-1)} \sum_\alpha \left[x_{i\alpha} - \sum_r (l_{ri} f_{r\alpha}) \right]^2 \quad (i = 1, 2, \ldots n), \tag{2}$$

$$\sum_\alpha \sum_q (l_{qi} f_{q\alpha} f_{r\alpha}) = \sum_\alpha (x_{i\alpha} f_{r\alpha}) \quad (i = 1, 2, \ldots n \text{ and } r = 1, 2, \ldots m), \tag{3}$$

$$\sum_i \sum_q \left(\frac{1}{s_i^2} l_{ri} l_{qi} f_{q\alpha} \right) = \sum_i \left(\frac{1}{s_i^2} l_{ri} x_{i\alpha} \right) \quad (\alpha = 1, 2, \ldots N \text{ and } r = 1, 2, \ldots m). \tag{4}$$

These equations are not all independent, however, since the result of multiplying both sides of (3) by $l_{pi}/s_i^2 (p = 1, 2, \ldots m)$ and summing over i is the same as that obtained by multiplying both sides of (4) by $f_{p\alpha}$ and summing over α. To obtain a unique solution we shall therefore impose m^2 additional conditions. Firstly, the scales of measurement for the m factors are entirely arbitrary, so that we may for convenience choose them so as to make

$$\sum_\alpha (f_{r\alpha}^2) = N - 1,$$

for all values of r. We may further suppose that the factors are such as to satisfy the "orthogonality" conditions

$$\sum_\alpha (f_{q\alpha} f_{r\alpha}) = 0,$$

$$\sum_i \left(\frac{l_{qi} l_{ri}}{s_i^2} \right) = 0,$$

for all values of q, r such that $q \neq r$.

Equation (2) can now be simplified, using also (3), and may be expressed in the form

$$s_i^2 = \frac{1}{(N-1)} \sum_\alpha (x_{i\alpha}^2) - \sum_r (l_{ri}^2).$$

If we let $\{a_{ij}\}$ denote the set of sample variances and covariances, so that

$$a_{ij} = \frac{1}{(N-1)} \sum_\alpha (x_{i\alpha} x_{j\alpha}),$$

then this equation becomes, even more simply,

$$s_i^2 = a_{ii} - \sum_r (l_{ri}^2). \tag{5}$$

Equations (3) and (4) may also be simplified by using matrix notation. Let X, L, F, and A denote the matrices whose typical elements are $x_{i\alpha}$, l_{ri}, $f_{r\alpha}$, and a_{ij} respectively; let V be the diagonal matrix whose

elements are the error variances $s_1^2, s_2^2, \ldots s_n^2$; and let J be the diagonal matrix $LV^{-1}L'$ (it will be diagonal in consequence of the last condition which we imposed above). Then equation (3) may now be written

$$L = \frac{1}{(N-1)} FX', \tag{6}$$

while equation (4) becomes

$$JF = LV^{-1}X,$$

or

$$F = J^{-1}LV^{-1}X. \tag{7}$$

Finally, by eliminating F from equations (6) and (7) and using the fact that $A = \frac{1}{(N-1)} XX'$, we find that

$$JL = LV^{-1}A. \tag{8}$$

3. It may be noted that the above equations give results which are independent of the units of measurement used, so that we may equally well use the correlation matrix R, with unit diagonal elements, in place of the covariance matrix A. Equation (7) is identical (except for the difference of notation) with that obtained by Bartlett (*loc. cit.*) for estimating a person's factors given the test loadings; the latter may in theory be found from equation (8). It does not, however, seem easy to find a satisfactory method of solving (8), since all the more obvious iterative procedures either do not in general converge, or else tend to unacceptable solutions in which one or more of the error variances vanish.

As an example we may consider the case where

$$A = \begin{bmatrix} 1{\cdot}000 & {\cdot}374 & {\cdot}560 & {\cdot}335 \\ {\cdot}374 & 1{\cdot}000 & {\cdot}219 & {\cdot}454 \\ {\cdot}560 & {\cdot}219 & 1{\cdot}000 & {\cdot}449 \\ {\cdot}335 & {\cdot}454 & {\cdot}449 & 1{\cdot}000 \end{bmatrix}.$$

If the presence of two general factors is assumed, it may be verified that equation (8) is satisfied (roughly) by

$$L = \begin{bmatrix} {\cdot}815 & {\cdot}608 & {\cdot}794 & {\cdot}725 \\ -{\cdot}354 & {\cdot}586 & -{\cdot}371 & {\cdot}477 \end{bmatrix},$$

$$V = [\ {\cdot}210 \quad {\cdot}287 \quad {\cdot}232 \quad {\cdot}247]$$

(the elements of V being for convenience printed in a row instead of diagonally); but no method has so far been found for arriving at this solution by successive approximations. It has been found possible to apply a method of this kind to the simple example given below, but this example is special in that the second and third tests have identical cor-

relations with the other two, which is apparently the only reason why the method works.

If we have

$$A = \begin{bmatrix} 1{\cdot}0 & {\cdot}4 & {\cdot}4 & {\cdot}2 \\ {\cdot}4 & 1{\cdot}0 & {\cdot}7 & {\cdot}3 \\ {\cdot}4 & {\cdot}7 & 1{\cdot}0 & {\cdot}3 \\ {\cdot}2 & {\cdot}3 & {\cdot}3 & 1{\cdot}0 \end{bmatrix},$$

and if we suppose the existence of one general factor, then the matrix of loadings L will consist simply of one row, and will satisfy the equation

$$hL = LV^{-1}A,$$

where

$$h = LV^{-1}L' \qquad (\text{and } h^2 = LV^{-1}AV^{-1}L').$$

As a first approximation to L we shall take the loadings given by Hotelling's first "principal component," namely

$$L_1 = [{\cdot}662 \quad {\cdot}857 \quad {\cdot}857 \quad {\cdot}540].$$

If we let V_1 denote the corresponding first approximation to V, then a second approximation to L is now given by

$$L_2 = \frac{1}{h_1} L_1 V_1^{-1} A,$$

where

$$h_1^2 = (L_1 V_1^{-1} A)(V_1^{-1} L_1').$$

The actual calculations are set out below.

L_1	·662	·857	·857	·540
V_1	·5618	·2656	·2656	·7084
$L_1V_1^{-1}$	1·1784	3·2267	3·2267	·7623
$L_1V_1^{-1}A$	3·9122	6·1854	6·1854	2·9340

$h_1^2 = 46{\cdot}7636 \qquad 1/h_1 = {\cdot}14623$

L_2	·5721	·9045	·9045	·4290.

The above process is then repeated until the required degree of approximation is reached, the successive approximations to L being as follows:—

·5721	·9045	·9045	·4290
·5058	·9177	·9177	·3772
·4845	·9198	·9198	·3634
·4801	·9201	·9201	·3607
·4793	·9202	·9202	·3602
·4791	·9202	·9202	·3601.

Thus we see that, correct to three figures, the test loadings are given by

$$L = [{\cdot}479 \quad {\cdot}920 \quad {\cdot}920 \quad {\cdot}360],$$

and the error variances by

$$V = [{\cdot}771 \quad {\cdot}153 \quad {\cdot}153 \quad {\cdot}870].$$

4. The above method of estimation will in future be referred to as "method II," in order to distinguish it from the previous method (Lawley, *loc. cit.*) which we shall call "method I."

If we continue to use the notation so far adopted, and if we allow σ_i^2 to represent not only the variance due to error but also the variance due to a possible factor specific to the i^{th} test, then the equation of estimation for method I may be written

$$L = LC^{-1}A, \tag{9}$$

where $$C = L'L + V.$$

Now by using the identity

$$(I + J)LC^{-1} = LV^{-1},$$

it is easily seen that equation (9) is equivalent to

$$JL = LV^{-1}(A - V); \tag{10}$$

and J is as before a diagonal matrix.

This equation is the same as (8), the corresponding equation for method II, except for the substitution of $(A - V)$ for A; but whereas we were unable to find a general iterative method for solving (8) we have been able to derive one for solving (10). The method given below may be regarded as superseding that previously suggested, which suffered from the disadvantage that it necessitated the calculation of the reciprocal of the correlation matrix, a somewhat laborious undertaking when the number of tests administered is large.

Suppose for convenience that just two general factors are assumed present, and let the two rows of L be denoted by L_p and L_q. Let L_{p1}, L_{q1}, and V_1 be first approximations to L_p, L_q, and V. Then second approximations are given by

$$\begin{aligned} L_{p2} &= \frac{1}{k_1} L_{p1} V_1^{-1}(A - V_1) \\ &= \frac{1}{k_1}\{L_{p1} V_1^{-1} A - L_{p1}\} \\ &= \frac{1}{k_1} P_1, \\ L_{q2} &= \frac{1}{k_1} L_{q1} V_1^{-1}(A - L'_{p2} L_{p2} - V_1) \\ &= \frac{1}{k_1}\{L_{q1} V_1^{-1} A - (L_{q1} V_1^{-1} L'_{p2}) L_{p2} - L_{q1}\} \\ &= \frac{1}{k_1} Q_1, \end{aligned}$$

where

$$k_1^2 = L_{p1} V_1^{-1} P_1', \qquad k_1^2 = L_{q1} V_1^{-1} Q_1'.$$

This process, repeated indefinitely, appears to converge in all cases.

When only one factor is to be estimated the calculations involved are extremely simple. Thus in the example given at the end of the preceding section, if we take as our approximation to L the loadings given by Spearman's method of estimating g, namely,

$$L_1 = [\cdot 496 \quad \cdot 823 \quad \cdot 823 \quad \cdot 374],$$

then the first stage of the calculation is as follows:—

L_1	·496	·823	·823	·374
V_1	·7540	·3227	·3227	·8601
$L_1 V_1^{-1}$	·6578	2·5504	2·5504	·4348
$L_1 V_1^{-1} A$	2·7851	4·7292	4·7292	2·0966
P_1	2·2891	3·9062	3·9062	1·7226
	$k_1^2 = 22{\cdot}1795$		$1/k_1 = {\cdot}21234$	
L_2	·4861	·8294	·8294	·3658.

Continuing the above process, we find that the approximations for L converge to

$$[\cdot 4807 \quad \cdot 8361 \quad \cdot 8361 \quad \cdot 3623].$$

Hence (as also obtained by the previous method of calculation), the loadings of the four tests are given, correct to three figures, by

$$L = [\cdot 481 \quad \cdot 836 \quad \cdot 836 \quad \cdot 362],$$

and the error (or specific) variances by

$$V = [\cdot 769 \quad \cdot 301 \quad \cdot 301 \quad \cdot 869].$$

5. It may be of interest to compare methods I and II with two other methods of factor estimation, namely, Hotelling's process of extracting "principal components" and the modification of it by Thomson (1934).

In Hotelling's process, if we suppose the tests to have been standardised (so that A is identical with the correlation matrix R), the matrix of loadings L satisfies the equation

$$KL = LA, \tag{11}$$

where

$$K = LL';$$

and in view of the fact that the factors satisfy an orthogonality condition, K is a diagonal matrix.

Comparing equations (8) and (11) we see that, formally, method II differs from Hotelling's only in having an extra weighting factor V^{-1},

though, as the example given at the end of section 3 shows, the differences between the results of the two processes may in practice be considerable. Identical results will be obtained only if the proportion of error variance to total variance is the same for each test, *i.e.* if the tests are all equally reliable. Hotelling's process is in fact given by the method of maximum likelihood when it is assumed from the start that this fact is so.

Thomson has stated his process only for the case where one general factor g is to be estimated, but it seems clear that this process can be generalised to estimate several factors at once, the equation of estimation being in this case

$$KL = L(A - V), \tag{12}$$

where again $K = LL'$ is a diagonal matrix. The actual calculations could be carried out by means similar to those employed for method I, the only difference being the omission of the weighting factor V^{-1}.

Thus method I differs from the generalisation of Thomson's process in that it gives greater weight to those tests whose error variances are proportionately small, and conversely. Both of these methods are such that when N, the number of individuals tested, tends to infinity the expected values of the correlation coefficients, as calculated from the factor loadings, tend to the corresponding population values; whereas in the other two methods considered this is not the case. It appears, however, that the results of methods I and II will tend to equality when n, the number of tests administered, becomes large compared with m, the number of factors to be estimated.

For comparison, the results obtained by all four methods in the example already considered are summarised below:

Method.	Loadings Obtained.			
Method I	·481	·836	·836	·362
Thomson	·486	·833	·833	·368
Method II	·479	·920	·920	·360
Hotelling	·662	·857	·857	·540

To sum up, it would seem that method I, used in conjunction with the equation *

$$F = LC^{-1}X = (I + J)^{-1}LV^{-1}X \tag{13}$$

for estimating an individual's factor measurements, has distinct advantages over other methods of estimation. The amount of calculation

* This equation is that given by Thomson's regression method of estimating a person's factors. The factors estimated by Bartlett's method would in this case, however, differ only in scale from those found by the regression method.

involved, using the new process put forward in this article, is not excessive, even when more than one factor is to be estimated. Furthermore, this method of estimation is almost the only one known to the writer for which a satisfactory test has been rigorously established for deciding whether a sufficient number of factors have been fitted. It may therefore be as well to restate this test, which is applicable when N is large, using the notation of the present article.

Let us make the hypothesis that there are exactly m general factors present, and let

$$w = \mathrm{N} \sum_{i<j} \left\{ \frac{(c_{ij} - a_{ij})^2}{s_i^2 s_j^2} \right\},$$

where as before $\{s_i^2\}$ represents the set of error (or specific) variances, and where

$$c_{ij} = \sum_{r=1}^{m} (l_{ri} l_{rj}).$$

Then, under the above hypothesis, w is distributed as χ^2 with p degrees of freedom, where

$$\begin{aligned} p &= \tfrac{1}{2}n(n-1) - mn + \tfrac{1}{2}m(m-1) \\ &= \tfrac{1}{2}\{(n-m)^2 - n - m\}. \end{aligned}$$

If a significantly high value of w is obtained, this will indicate that we must reject the hypothesis and assume the existence of more than m factors.

6. It may of course happen that we are provided with independent estimates of the error variances and covariances (it is not then necessary to assume that the errors are uncorrelated). Thus, for instance, the n tests may all have been given more than once, in parallel forms, to each of the N individuals. In that case it is possible to carry out an analysis of variance and covariance and obtain two independent sets of estimated variances and covariances, one set being due to differences between persons while the other is due only to error.

Let these two sets be denoted by $\{a_{ij}\}$ and $\{b_{ij}\}$ respectively, and let A and B be the matrices of which a_{ij} and b_{ij} are the typical elements. Then the equation of estimation for the factor loadings will in this case be

$$\mathrm{HL} = \mathrm{LB}^{-1}\mathrm{A}, \tag{14}$$

where

$$\mathrm{H} = \mathrm{LB}^{-1}\mathrm{L}'$$

is a diagonal matrix.

In the solution of this equation the rows of L are proportional to the latent vectors of $\mathrm{B}^{-1}\mathrm{A}$, and the elements of H are the corresponding

latent roots. The individual factor measurements are given in terms of the test scores by the equation

$$F = H^{-1}LB^{-1}X = LA^{-1}X. \qquad (15)$$

This method of procedure represents nothing new, however, since it is equivalent to the "discriminant function" analysis put forward by Fisher (1938, 1939).

SUMMARY.

7. A method of factor estimation is given in which assumptions are made only about the form of the error distributions of the tests administered. This is compared with a method previously suggested in which, on the contrary, the test scores and the individual factor measurements were assumed to be normally distributed over the population of individuals tested. A comparison is also made with other processes at present in use.

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XV.—On the Estimation of Statistical Parameters.

By **A. C. Aitken,** D.Sc., F.R.S., Mathematical Institute, University of Edinburgh, and **H. Silverstone,** Ph.D., New Zealand.

(MS. received December 30, 1941. Read February 2, 1942.)

1. Introductory.

The present paper communicates some of the results given in a thesis submitted by Mr Silverstone to the University of Edinburgh in May 1939. The starting-point, the adoption of the postulates of unbiased estimate and minimum sampling variance, and the analogies which would emerge with the theories of maximum likelihood and of linear estimation by Least Squares, were suggested by the senior author. Thanks are due to Dr R. P. Gillespie for some helpful preliminary discussion in 1937 and 1938.

The senior author is also responsible for the composition of this paper and for modifications of the original material. Owing to the hazards and delays of correspondence arising from the present war, it has not been possible for the authors, one living in Scotland and the other in New Zealand, to continue their collaboration.

2. Maximum Probability Density and Minimum Variance.

In the theory of linear estimation by Least Squares from linear observational equations affected by error, the normal equations for the optimal values of the unknowns can be deduced by two quite different sets of postulates. The first, (A), proceeds by (i) assuming a normal distribution of errors of observation, and (ii) accepting as optimal values of the unknowns those which make the compound probability density a maximum; the second, (B), proceeds by (i) assuming that the optimal values are unbiased linear combinations of the observations, and (ii) accepting those particular linear combinations for which the compound variance of error is a minimum. It is to be noted that in (B) the law of error of the observations is not restricted to be of normal or of any assigned type, and could indeed be different for each observation.

In the theory of estimation of statistical parameters the principle of maximum likelihood (Fisher, 1921) bears a certain resemblance to postulate

(ii) of the first set (A) of postulates of Least Squares. The principle accepts, in fact, as best estimate of an unknown parameter θ in a probability function $\phi(x; \theta)$ of assigned type, that function $t(x_1, x_2, \ldots, x_n)$ of the sample values x_j which makes the compound probability density of the x_j a maximum. The vindication of the principle is largely *a posteriori*, arising from the circumstance that if the estimating function t has a sampling distribution of normal type, then, within the class of unbiased analytic functions estimating θ, t has also the least possible variance. Even when the distribution of t is not normal, it tends more and more to become so as the size of sample is increased, and the property of minimum variance tends to hold with closer and closer approximation.

This analogy with the position in Least Squares suggests a basis for estimation alternative to that of maximum likelihood and closely resembling the postulates (B). Let us adopt postulates (i) of unbiased or consistent estimate, (ii) of minimum variance of estimating function. The investigation of the consequences of these postulates sets the principle of maximum likelihood in an interesting light. For example, simple conditions emerge under which maximum likelihood provides an estimate accurately possessing minimum variance, even though the sample is finite and the distribution of the estimating function is not normal. At the same time the actual value of the minimum variance is obtained with special ease.

The arbitrariness of the postulates needs no emphasis. They are inevitably arbitrary; but so are those of any proposed alternative set. The restriction to a linear operation in the condition of consistency is admittedly a severe one. One is free to choose a condition of consistency different from the one here adopted, but the subsequent analysis will be found decidedly less tractable, perhaps impossible. Again, since, as is well known, many plausibly posed questions in the Calculus of Variations do not admit of analytic solution, it evokes no surprise that wide classes of probability functions are not amenable to estimation of parameters by the suggested postulates.

3. Postulates for Estimation of Parameters.

Let $\phi(x; \theta)$ be a probability function containing an unknown parameter θ, which is to be estimated from a sample $x_1, x_2, \ldots, x_n$ of n values of x. The range of x is presumed to be independent of θ; and ϕ is presumed to be uniformly differentiable with respect to θ. Let us denote the compound probability density of the sample by

$$\Phi(x_1, x_2, \ldots, x_n; \theta) \equiv \Phi.$$

It is required to estimate θ by a function $t(x_1, x_2, \ldots, x_n) \equiv t$, where t is independent of θ. Let us adopt the two following postulates:—

(i) That t shall be an unbiased or consistent estimate of θ, in the sense that the mean value or expectation of t over all n-ary samples $x_1, x_2, \ldots, x_n$ shall be θ; that is,

$$\iiint \ldots t\Phi dx_1 dx_2 \ldots dx_n = \theta. \tag{1}$$

(ii) That the sampling variance of t over all such samples shall be a minimum; that is,

$$\iiint \ldots (t-\theta)^2 \Phi dx_1 dx_2 \ldots dx_n = \text{minimum}. \tag{2}$$

We have also the necessary condition of total probability, namely,

(iii)
$$\iiint \ldots \Phi dx_1 dx_2 \ldots dx_n = 1. \tag{3}$$

The problem is thus a minimal problem in the Calculus of Variations, of positive definite type and formally simple. Let us write the Euler equation for its solution in the form

$$(t-\theta)\Phi - \lambda(\theta)\Phi_\theta = 0, \quad \text{where} \quad \Phi_\theta = \frac{\partial}{\partial\theta}\Phi, \tag{4}$$

the Lagrange function $\lambda(\theta)$ being independent of $x_1, x_2, \ldots, x_n$. We then have

$$\begin{aligned} t &= \theta + \lambda\Phi_\theta/\Phi \\ &= \theta + \lambda\left(\frac{\partial}{\partial\theta}\log\Phi\right), \end{aligned} \tag{5}$$

where t must be independent of θ. In short, an estimating function t exists, provided that the derivative of $\log\Phi$ with respect to θ can be resolved as follows:

$$\frac{\partial}{\partial\theta}\log\Phi = (t-\theta)/\lambda(\theta). \tag{6}$$

Two standard examples will serve to illustrate

Example 1.—To estimate μ in $\phi(x;\ \mu) \equiv (2\pi)^{-\frac{1}{2}} \exp -\frac{1}{2}(x-\mu)^2$.

Here
$$\Phi = (2\pi)^{-\frac{1}{2}n} \exp\{-\tfrac{1}{2}\textstyle\sum(x_j-\mu)^2\},$$

$$\begin{aligned} \frac{\partial}{\partial\mu}\log\Phi &= \textstyle\sum x_j - n\mu \\ &= (t-\mu)/\lambda(\mu) \end{aligned}$$

provided that $\lambda(\mu) = 1/n$, in which case $t = \sum x_j/n = \bar{x}$. Thus the estimate of μ is $\bar{x}$, the mean of the observations. As for λ, let us anticipate the general result by noting that its value here, $1/n$, is the minimum variance in question, namely, the sampling variance of $\bar{x}$.

Example 2.—To estimate σ^2 in $(2\pi\sigma^2)^{-\frac{1}{2}} \exp -\frac{1}{2}x^2/\sigma^2$.

Here

$$\frac{\partial}{\partial\sigma^2}\log\Phi = -\frac{n}{2\sigma^2} + \frac{1}{2}\frac{\Sigma x_j^2}{\sigma^4}$$
$$= (t-\sigma^2)/\lambda(\sigma^2)$$

provided that $\lambda = 2\sigma^4/n$, in which case $t = \sum x_j^2/n$. Again, $2\sigma^4/n$ is the variance of the estimate t of σ^2.

If we had had to estimate both μ and σ^2 in

$$\phi(x;\ \mu, \sigma^2) \equiv (2\pi\sigma^2)^{-\frac{1}{2}} \exp\{-\tfrac{1}{2}(x-\mu)^2/\sigma^2\}$$

we should have proceeded by first estimating μ. The estimate proves as before to be the mean $\bar{x} = \sum x_j/n$, with variance $\lambda = \sigma^2/n$. This estimate being accepted, we know from standard results that the compound probability density *relative to the estimate* $\bar{x}$, not to the unknown μ, is

$$\Phi(x_1, x_2, \ldots, x_n;\ \bar{x}, \sigma^2) = c(\sigma^2)^{-\frac{1}{2}(n-1)} \exp\{-\tfrac{1}{2}\sum(x_j-\bar{x})^2/\sigma^2\}.$$

From this, estimating anew for σ^2 and writing the customary s^2 instead of t, we find

$$s^2 = \sum(x_j-\bar{x})^2/(n-1) \quad \text{and} \quad \lambda = 2\sigma^4/(n-1),$$

the latter being the well-known sampling variance of s^2. The sampling distribution of s^2 is known to be not normal, but of Gamma Type.

As a further point of interest, let us try to estimate, not σ^2 but σ in

$$\phi(x;\ \sigma) = (2\pi\sigma^2)^{-\frac{1}{2}} \exp\ -\tfrac{1}{2}x^2/\sigma^2.$$

We obtain

$$\frac{\partial}{\partial\sigma}\log\Phi = -n/\sigma + \sum x_j^2/\sigma^3.$$

Evidently this cannot be put in the form $(t-\sigma)/\lambda(\sigma)$, where t is independent of σ, and λ is independent of the x_j. In this example, therefore, we may not estimate for σ, but must estimate for σ^2. The reason is that the property of minimum variance is not preserved under non-linear transformations of the parameter. It follows that in most cases we must estimate, not θ but some function $\tau(\theta)$. What function τ is will appear in the sequel.

Example 3.—The Cauchy distribution

$$\phi(x;\ \mu) = \{1+(x-\mu)^2\}/\pi.$$

Let us try to estimate μ, the abscissa of symmetry. We have

$$\frac{\partial}{\partial\mu}\log\Phi = 2\sum(x_j-\mu)/\{1+(x_j-\mu)^2\}.$$

This cannot be put into the form $(t-\theta)/\lambda(\theta)$. The Cauchy distribution is thus intractable by the present method.

Example 4.—

$$\phi(x;\ \theta) = \{\Gamma(k)\}^{-1}\theta^{-k}x^{k-1}e^{-x/\theta}$$

$$\frac{\partial}{\partial\theta}\log\Phi = -nk\theta^{-1} + \sum x_j/\theta^2$$

$$= (t-\theta)/\lambda(\theta)$$

provided that $t = \sum x_j/nk$, and $\lambda = \theta^2/nk$.

Here let us observe that the sample is finite and that the distribution of t is non-normal.

4. Comparison with Maximum Likelihood.

It appears, then, that an estimating function $\tau(\theta)$ exists when $\partial \log \Phi/\partial\theta$ resolves itself into the form

$$\frac{\partial}{\partial\theta}\log\Phi = \frac{\partial\tau(\theta)}{\partial\theta}\{t(x_1, x_2, \ldots, x_n) - \tau(\theta)\}/\lambda(\theta). \tag{1}$$

We estimate $\tau(\theta)$ by t, and deduce the estimate of θ. Thus our procedure, when possible, consists in virtually equating $t(x_1, x_2, \ldots, x_n)$ to $\tau(\theta)$, indeed in formally solving the equation in θ,

$$\frac{\partial}{\partial\theta}\log\Phi = 0. \tag{2}$$

Now this is precisely what is done in R. A. Fisher's method (Fisher, 1921) of maximum likelihood, for $\log\Phi$ is the logarithm of the compound probability density of the sample; but in that case it is done without consideration of the existence or nature of $\lambda(\theta)$, a question which does not arise. From our standpoint, on the other hand, the existence of $\lambda(\theta)$ is fundamental; for when $\lambda(\theta)$ exists, the estimate by maximum likelihood (of $\tau(\theta)$, that is, or the trivial linear function $a\tau(\theta)+b$, but not of non-linear functions of $\tau(\theta)$) has also minimum variance, even in the case of a finite sample.

5. Variance of Estimate.

The minimal variance v of $\tau(\theta)$, that is the sampling variance of the estimate, can be expressed in several different ways. To avoid prolixity, let us suppose that the suitable parameter has been ascertained, and is henceforth called θ. We have then

$$v = \iiint \ldots (t-\theta)^2\Phi\, dx_1 dx_2 \ldots dx_n, \tag{1}$$

where

$$t-\theta=\lambda(\theta)\frac{\partial}{\partial\theta}\log\Phi.$$

Hence

$$v=\lambda^2\iiint\ldots(\Phi_\theta/\Phi)^2\Phi dx_1dx_2\ldots dx_n. \tag{2}$$

Now

$$\frac{\partial^2}{\partial\theta^2}\log\Phi=\Phi^{-1}\frac{\partial^2\Phi}{\partial\theta^2}-(\Phi_\theta/\Phi)^2$$

whence

$$(\Phi_\theta/\Phi)^2=\frac{\partial^2\Phi}{\partial\theta^2}\Phi^{-1}-\left(\frac{\partial^2}{\partial\theta^2}\log\Phi\right).$$

Hence, since the integrals of Φ_θ and $\Phi_{\theta\theta}$, the range being independent of θ, are both zero, we have

$$v=-\lambda^2\iiint\ldots\left(\frac{\partial^2}{\partial\theta^2}\log\Phi\right)\Phi dx_1dx_2\ldots dx_n. \tag{3}$$

But again, since

$$\frac{\partial}{\partial\theta}\log\Phi=(t-\theta)/\lambda(\theta),\quad \frac{\partial^2}{\partial\theta^2}\log\Phi=(t-\theta)\frac{\partial}{\partial\theta}\{\lambda(\theta)\}^{-1}-\{\lambda(\theta)\}^{-1},$$

we have

$$v=-\lambda^2\iiint\ldots(-\lambda^{-1}\Phi)dx_1dx_2\ldots dx_n, \tag{4}$$

since

$$\iiint\ldots(t-\theta)\Phi dx_1dx_2\ldots dx_n=0.$$

Hence $v=\lambda$, while equally we see from (3) that v^{-1} is the mean value of $-\partial^2L/\partial\theta^2$, where $L=\log\Phi$. This last is a well-known result of R. A. Fisher, which here we see holding accurately in finite samples, provided that $\lambda(\theta)$ exists, and that $\partial L/\partial\theta=(t-\theta)/\lambda(\theta)$.

It is easy to prove from $v=\lambda$ that the only Pearsonian distribution for which the mean is a sufficient statistic for locating the curve is the normal distribution. For since the variance of the mean of *any* distribution is σ^2/n, we must have

$$\frac{\partial}{\partial\theta}\log\Phi=n(\theta-\sum x/n)/\sigma^2$$
$$=(n\theta-\sum x)/\sigma^2,$$

whence

$$\frac{\partial}{\partial\theta}\log\phi=-(x-\theta)/\sigma^2$$

and so

$$\phi=c(x,\sigma)\exp\{-\tfrac{1}{2}(x-\theta)^2/\sigma^2\}.$$

There is an arbitrary step here, namely, the choice of the arbitrary function c, which may involve x and σ but not θ, but a census of the Pearsonian curves will show that no other Pearsonian type except the normal will satisfy the problem. A probability function of Type A would, however, be admissible.

6. Probability Functions Amenable to the Postulates.

By integrating the fundamental equation of estimation

$$\frac{\partial}{\partial\theta}\log\Phi=(t-\theta)/\lambda(\theta) \tag{1}$$

we may ascertain what class of probability function is amenable to the present method. Since t is independent of θ, and λ is independent of the x_j, we have

$$\log\Phi=\mu(\theta)(t-\theta)+\nu(\theta)+C, \tag{2}$$

where

$$\mu(\theta)=\int_0^\theta\frac{d\theta}{\lambda(\theta)},\qquad \nu(\theta)=-\int_0^\theta\frac{\theta d\theta}{\lambda(\theta)}, \tag{3}$$

and where C is a function of the x_j but not of θ. Thus $\phi(x;\ \theta)$ must be of the form

$$\phi(x;\ \theta)=\exp\{\mu(\theta)t(x)+F(\theta)+f(x)\} \tag{4}$$

and must also satisfy the condition of total probability, § 3 (3). This class of functions includes the normal function and functions of Pearson's Type III.

By means of the above result we are able to determine what function $\tau(\theta)$ should be estimated, if necessary, instead of θ. For if

$$\phi=\exp(\mu t+F+f)$$

then

$$\Phi=\exp(\mu\textstyle\sum t+nF+\sum f), \tag{5}$$

and so

$$\frac{\partial}{\partial\tau}\log\Phi=\frac{\partial\mu}{\partial\tau}\sum t+n\frac{\partial F}{\partial\tau}$$

$$=n\frac{\partial\mu}{\partial\tau}\left(\sum t/n+\frac{\partial F}{\partial\mu}\right).$$

Hence if

$$\tau=-\partial F/\partial\mu=-\frac{\partial F}{\partial\theta}\Big/\frac{\partial\mu}{\partial\theta},$$

we arrive at

$$\frac{\partial}{\partial\tau}\log\Phi=(\textstyle\sum t/n-\tau)/\lambda(\tau). \tag{6}$$

Thus we estimate for $\tau(\theta)$, as given by $-\partial F/\partial\mu$, and our estimate is indeed $\sum t/n$, its variance being $(n\partial\mu/\partial\tau)^{-1}$.

Example.—

$$\phi(x;\ \sigma)=(2\pi\sigma^2)^{-\frac{1}{2}}\exp\ -\tfrac{1}{2}x^2/\sigma^2.$$

Here

$$\mu=-1/2\sigma^2, \qquad F=-\log\sigma,$$

whence

$$-\frac{\partial F}{\partial\mu}=-\frac{\partial F}{\partial\sigma}\Big/\frac{\partial\mu}{\partial\sigma}=\sigma^2,$$

which we must therefore estimate. Also $v=\left(n\dfrac{\partial\mu}{\partial\tau}\right)^{-1}=2\sigma^4/n$.

Or again, purely in terms of μ and F, we might have written

$$v^{-1}=n\frac{\partial\mu}{\partial\tau}=n\frac{\partial\mu}{\partial\theta}\Big/\frac{\partial\tau}{\partial\theta}$$

$$=-n\frac{\partial\mu}{\partial\theta}\Big/\left(\frac{\partial^2F}{\partial\mu^2}\cdot\frac{\partial\mu}{\partial\theta}\right), \qquad \left(\text{since } \tau=-\frac{\partial F}{\partial\mu}\right)$$

whence

$$v=-n^{-1}\frac{\partial^2F}{\partial\mu^2}.$$

7. Sufficient Statistics and Minimum Variance.

In a paper, "On distributions admitting a sufficient statistic," B. O. Koopman finds the general form of a distribution function (Koopman, 1936) admitting the determination of sufficient statistics for the estimation of some or all of the parameters θ_j.

The intuitive notion of sufficiency is this, that a statistic should use up the whole of the relevant information contained in a sample. The formulation of this idea with respect to $\phi(x;\ \theta)$ leads to a sufficient statistic estimating θ, provided that the equality

$$\frac{\Phi(x_1, x_2, \ldots, x_n;\ \theta)}{\Phi(x_1', x_2', \ldots, x_n';\ \theta)}=\frac{\Phi(x_1, x_2, \ldots, x_n\ ;\ \theta')}{\Phi(x_1', x_2', \ldots, x_n';\ \theta')}$$

is implied by $t(x_1', x_2', \ldots, x_n')=t(x_1, x_2, \ldots, x_n)$ for any two values θ and θ' of the parameter and any two possible samples, as indicated.

Koopman finds that, if ϕ is analytic and non-zero over some continuous range of θ, then

$$\phi(x;\ \theta)=\exp\{F_1(\theta)f_1(x)+F_2(\theta)+f_2(x)\},$$

where the functions in the exponent are real, single-valued, analytic functions of their arguments.

Now this is the form which we obtained in § 6 for probability functions amenable to the postulates of unbiased estimate and minimum variance.

We see, therefore, that these functions also admit sufficient statistics for the estimation of the parameters concerned.

8. Simultaneous Estimation of Parameters.

Problems of so-called *simultaneous* estimation lead to the following equations, in k parameters:

$$t_j(x_1, x_2, \ldots, x_n) = \theta_j - \lambda_j(\theta_1, \theta_2, \ldots, \theta_k)\frac{\partial}{\partial\theta_j}\log\Phi, \qquad j = 1, 2, \ldots, k,$$

where $t_1, t_2, \ldots, t_k$ are functions of $x_1, x_2, \ldots, x_k$ independent of $\theta_1, \theta_2, \ldots, \theta_k$ and $\lambda_1, \lambda_2, \ldots, \lambda_k$ are functions of $\theta_1, \theta_2, \ldots, \theta_k$ independent of $x_1, x_2, \ldots, x_n$. The strictly simultaneous solution of these equations is usually difficult. It is, indeed, extremely rare to find all the parameters admitting equations from which the factor $\lambda(\theta)$ is extricable. As we have seen, even the estimation of μ and σ^2 from a normal sample involves us in difficulty, unless we agree to estimate, first μ by $\bar{x}$, and then σ^2 from the corresponding modified probability density, relative to $\bar{x}$, not to μ. The estimation of the means μ_{10} and μ_{01} of x and y, the variances μ_{20} and μ_{02} and the correlation coefficient ρ from a bivariate normal sample of n paired values (x_j, y_j) offers still greater difficulties.

It is proposed to reserve the subject of simultaneous estimation and other aspects of the method of minimum variance for future consideration.

Summary.

In the problem of estimating from sample the value of a parameter in a probability function new postulates are suggested of unbiased linear estimate and minimum sampling variance. A comparison is made, with illustrative examples, between this method and the principle of maximum likelihood, and ground common to the two is traversed. The new postulates are also placed in relation to the theory of sufficient statistics.

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Cells, Liver,—Intra-cellular Canaliculi in,
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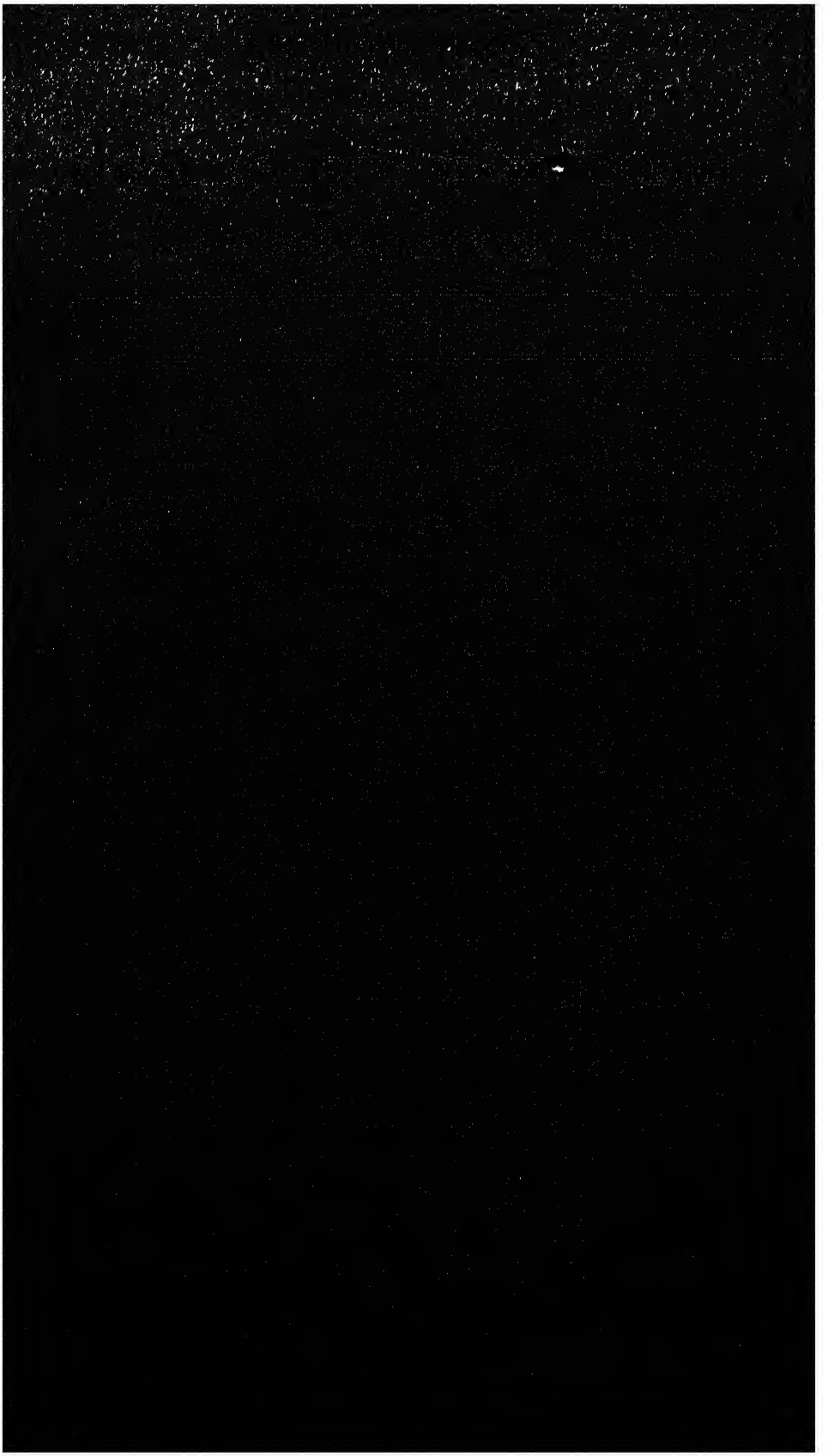

VIII.—The Central Limit Theorem for a Convergent Non-homogeneous Finite Markov Chain.*

By **J. L. Mott**, Department of Mathematics, University of Edinburgh. *Communicated by* Professor A. C. AITKEN, F.R.S.

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SYNOPSIS

The distribution of x_n, the number of occurrences of a given one of k possible states of a non-homogeneous Markov chain $\{P_j\}$ in n successive trials, is considered. It is shown that if $P_n \to P$, a positive-regular stochastic matrix, as $n \to \infty$ then the distribution about its mean of $x_n/n^{\frac{1}{2}}$ tends to normality, and that the variance tends to that of the corresponding distribution associated with the homogeneous chain $\{P\}$.

1. INTRODUCTION

WE say that a non-homogeneous finite Markov chain is *convergent* if the sequence $\{P_j\}$ of matrices of one-step transition probabilities of the chain tends to a limit matrix P as $j \to \infty$, and if P is *positive-regular* so that $\lim_{n\to\infty} P^n = U$ exists and the elements of U are all positive. We know that if this is so the product $P_1P_2 \ldots P_n = P^{(n)}$ also tends to U as $n \to \infty$ (Mott 1957, p. 379).

We can interpret this result in a second way. Let x_n be the number of occurrences in n successive trials of a given one of the k states; we can take this state to be the first without loss of generality. Then x_n has a certain probability distribution whose mean $\bar{x}_n$ is given by

$$\bar{x}_n = \underline{a}\{P^{(1)} + P^{(2)} + \ldots + P^{(n)}\}[1, \cdot, \ldots, \cdot]',$$

where the row vector $\underline{a}$ denotes the initial probability distribution and $[1, \cdot, \ldots, \cdot]'$ denotes the transpose of the row vector $[1, \cdot, \ldots, \cdot]$. Thus if $P^{(n)} \to U$, as is the case above, then $\bar{x}_n \to n\underline{a}U[1, \cdot, \ldots, \cdot]'$, so that the mean frequency of occurrences of the given state in the non-homogeneous chain of trials tends to asymptotic equality with the mean frequency of occurrence in the corresponding homogeneous chain.

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We can consider likewise the higher moments of the distribution of x_n, or of $x_n/n^{\frac{1}{2}}$. We now show that all moments of the distribution of $x_n/n^{\frac{1}{2}}$ in a sequence of n successive trials of the convergent non-homogeneous chain tend to asymptotic equality to the corresponding moments for the associated homogeneous chain. Since the distribution of $x_n/n^{\frac{1}{2}}$ in the latter case tends to normality as $n \to \infty$ (Fréchet 1938, p. 160) it follows that the distribution of $x_n/n^{\frac{1}{2}}$ also tends to normality in the case of a convergent non-homogeneous chain, and that the variances of two such associated distributions are asymptotically equal.

2. The Moments of the Distribution of x_n

The nature of the proof of these results in the general case of a chain with k states is exemplified by the case of $k=3$, and for clarity we shall set out the details of the proof for this particular case. We write $\mu_r^{(n)}$ for the rth moment, about the current mean, of the distribution of x_n. We show that the difference between the values of $\mu_2^{(n)}/n$ for the non-homogeneous convergent and associated homogeneous chains tends to zero as $n \to \infty$, and then that

$$\mu_{2m}^{(n)} = (2m-1)(2m-3) \ldots 3\{\mu_2^{(n)}\}^2 + O(n^{m-1}) \tag{1}$$

and

$$\mu_{2m+1}^{(n)} = O(n^m). \tag{2}$$

The result that the distribution of $x_n/n^{\frac{1}{2}}$ is asymptotically normal then follows.

We consider the increments to the moments on passing from one stage to the next and hence obtain the moments of the distribution after n trials by the summation of n such increments. Let the initial probabilities of the occurrence of the three states be given by the elements of the row vector $[\alpha, \beta, \gamma]$, and denote a typical matrix P_j of the chain by $P_j \equiv [{}_jp_{ab}]$ for $a, b = 1, 2, 3$. Then the characteristic function (c.f.) of the distribution of x_n is

$$[\alpha e^{it}, \beta, \gamma] e^{-\bar{x}_n it} \left\{ \prod_{j=1}^{n} \begin{bmatrix} {}_jp_{11}e^{it} & {}_jp_{12} & {}_jp_{13} \\ {}_jp_{21}e^{it} & {}_jp_{22} & {}_jp_{23} \\ {}_jp_{31}e^{it} & {}_jp_{32} & {}_jp_{33} \end{bmatrix} \right\} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

if, as always, we refer the distribution to its current mean $\bar{x}_n$ as origin.

To bring into prominence the moments of the distribution and to utilize the particular nature of stochastic matrices, namely, that their row

sums are unity, it is convenient to make a certain matrix transformation: * we replace each matrix ( ) in the c.f. by $H(\ )H^{-1}$, where

$$H=\begin{bmatrix} 1 & \cdot & \cdot \\ -1 & 1 & \cdot \\ -1 & \cdot & 1 \end{bmatrix} \quad \text{and so} \quad H^{-1}=\begin{bmatrix} 1 & \cdot & \cdot \\ 1 & 1 & \cdot \\ 1 & \cdot & 1 \end{bmatrix}.$$

Under this transformation a typical matrix occurring in the c.f. becomes

$$P_j(t)\equiv\begin{bmatrix} {}_jp_{11}e^{it}+{}_jp_{12}+{}_jp_{13}, & {}_jp_{12}, & {}_jp_{13} \\ ({}_jp_{21}-{}_jp_{11})e^{it}+({}_jp_{22}-{}_jp_{12})+({}_jp_{23}-{}_jp_{13}), & {}_jp_{22}-{}_jp_{12}, & {}_jp_{23}-{}_jp_{13} \\ ({}_jp_{31}-{}_jp_{11})e^{it}+({}_jp_{32}-{}_jp_{12})+({}_jp_{33}-{}_jp_{13}), & {}_jp_{32}-{}_jp_{12}, & {}_jp_{33}-{}_jp_{13} \end{bmatrix},$$

and the c.f. of the distribution of x_n is

$$\phi_n(t)\equiv e^{-\bar{x}_n it}[\alpha e^{it},\ \beta,\ \gamma]H^{-1}\left\{\prod_{j=1}^{n}P_j(t)\right\}H[1,\ 1,\ 1]'.$$

Thus

$$\begin{aligned}\phi_n(t)&\equiv e^{-\bar{x}_n it}[\alpha e^{it}+\beta+\gamma,\ \beta,\ \gamma]\left\{\prod_{j=1}^{n}P_j(t)\right\}[1,\cdot,\cdot]' \\ &=\underline{a}_n[1,\cdot,\cdot]'\end{aligned}$$

say, where $\underline{a}_n=[\alpha_n,\ \beta_n,\ \gamma_n]$. We can expand the exponential terms as series in ascending powers of (it) and so write $\phi_n(t)$ in the form

$$\phi_n(t)\equiv\left[1+\mu_2^{(n)}\frac{(it)^2}{2!}+\ldots,\quad \nu_0^{(n)}+\nu_1^{(n)}(it)+\nu_2^{(n)}\frac{(it)^2}{2!}+\ldots,\quad \omega_0^{(n)}+\omega_1^{(n)}(it)+\ldots,\right][1,\cdot,\cdot]'.$$

We have used the same notation in α_n as for the moments; this is because the effect of the post-multiplication of $\underline{a}_n$ by the column vector $[1,\cdot,\cdot]'$ is to give $\phi_n(t)=\alpha_n$, and this fact accounts also for the absence of a term $\mu_1^{(n)}(it)$ which does not occur since the distribution is referred to its current mean as origin.

For finding the increments to the moments we are led to the study of the corresponding change in $\phi_j(t)$ on transition from the jth to the $(j+1)$th stage, and so to the study of the change in α_j. This necessitates in turn the study of $\underline{a}_j$, for the vectors $\underline{a}_j$ and $\underline{a}_{j+1}$ are related by the recurrence relation

$$\underline{a}_{j+1}=e^{-\Delta_j it}\underline{a}_jP_{j+1}(t), \tag{3}$$

where Δ_j is the increment to the mean on passing from the jth to the $(j+1)$th stage.

* I am indebted to Professor A. C. Aitken for this suggestion.

3. The Variance of the Distribution of x_n

For conciseness we write

$$_j\delta_{ab} = {}_j p_{ab} - {}_j p_{1b} \quad \text{for} \quad a = 2, 3 \quad \text{and} \quad b = 1, 2, 3,$$

and we then have on comparison of the elements of the first column of (3) that

$$\begin{aligned}
a_{j+1} &= \{1 - \Delta_j(it) + \Delta_j^2(it)^2/2! - \dots\}\{1 + [{}_{j+1}p_{11} + {}_{j+1}\delta_{21}\nu_0^{(j)} + {}_{j+1}\delta_{31}\omega_0^{(j)}](it) \\
&\quad + [\mu_2^{(j)} + {}_{j+1}p_{11} + 2_{j+1}\delta_{21}\nu_1^{(j)} + 2_{j+1}\delta_{31}\omega_1^{(j)} + {}_{j+1}\delta_{21}\nu_0^{(j)} + {}_{j+1}\delta_{31}\omega_0^{(j)}](it)^2/2! + \dots \\
&= 1 + [{}_{j+1}p_{11} + {}_{j+1}\delta_{21}\nu_0^{(j)} + {}_{j+1}\delta_{31}\omega_0^{(j)} - \Delta_j](it) + \{[\mu_2^{(j)} + {}_{j+1}p_{11} + 2_{j+1}\delta_{21}\nu_1^{(j)} \\
&\quad + 2_{j+1}\delta_{31}\omega_1^{(j)} + {}_{j+1}\delta_{21}\nu_0^{(j)} + {}_{j+1}\delta_{31}\omega_0^{(j)}] - 2\Delta_j[{}_{j+1}p_{11} + {}_{j+1}\delta_{21}\nu_0^{(j)} + {}_{j+1}\delta_{31}\omega_0^{(j)}] \\
&\quad + \Delta_j^2\}(it)^2/2! + \dots
\end{aligned}$$

The coefficient of (it) here is necessarily zero so that

$$\Delta_j = {}_{j+1}p_{11} + {}_{j+1}\delta_{21}\nu_0^{(j)} + {}_{j+1}\delta_{31}\omega_0^{(j)}. \tag{4}$$

Using this result we have that $\Delta\mu_2^{(j)}$, the increment to $\mu_2^{(j)}$ on transition from the jth to the $(j+1)$th stage, is given by

$$\Delta\mu_2^{(j)} + 2_{j+1}\delta_{21}\nu_1^{(j)} + 2_{j+1}\delta_{31}\omega_1^{(j)} + \Delta_j - \Delta_j^2. \tag{5}$$

We now need to find expressions for $\nu_1^{(n)}$ and $\omega_1^{(n)}$, and therefore we compare the second, and the third, columns of the vector recurrence relation (3). From the second column we find

$$\begin{aligned}
\beta_{j+1} &= \{1 - \Delta_j(it) + \dots\}\{({}_{j+1}p_{12} + {}_{j+1}\delta_{22}\nu_0^{(j)} + {}_{j+1}\delta_{32}\omega_0^{(j)}) + ({}_{j+1}\delta_{22}\nu_1^{(j)} + {}_{j+1}\delta_{32}\omega_1^{(j)})(it) \\
&\qquad + \dots\} \\
&= ({}_{j+1}p_{12} + {}_{j+1}\delta_{22}\nu_0^{(j)} + {}_{j+1}\delta_{32}\omega_0^{(j)}) + [({}_{j+1}\delta_{22}\nu_1^{(j)} + {}_{j+1}\delta_{32}\omega_1^{(j)}) - \Delta_j({}_{j+1}p_{12} + {}_{j+1}\delta_{22}\nu_0^{(j)} \\
&\qquad + {}_{j+1}\delta_{32}\omega_0^{(j)})](it) + \dots,
\end{aligned}$$

from which we have

$$\nu_0^{(j+1)} = {}_{j+1}p_{12} + {}_{j+1}\delta_{22}\nu_0^{(j)} + {}_{j+1}\delta_{32}\omega_0^{(j)} \tag{6}$$

and

$$\begin{aligned}
\nu_1^{(j+1)} &= {}_{j+1}\delta_{22}\nu_1^{(j)} + {}_{j+1}\delta_{32}\omega_1^{(j)} - ({}_{j+1}p_{12} + {}_{j+1}\delta_{22}\nu_0^{(j)} + {}_{j+1}\delta_{32}\omega_0^{(j)})\Delta_j \\
&= {}_{j+1}\delta_{22}\nu_1^{(j)} + {}_{j+1}\delta_{32}\omega_1^{(j)} - \nu_0^{(j+1)}\Delta_j.
\end{aligned} \tag{7}$$

From the third columns we find similarly

$$\omega_0^{(j+1)} = {}_{j+1}p_{13} + {}_{j+1}\delta_{23}\nu_0^{(j)} + {}_{j+1}\delta_{33}\omega_0^{(j)}$$

and

$$\omega_1^{(j+1)} = {}_{j+1}\delta_{23}\nu_1^{(j)} + {}_{j+1}\delta_{33}\omega_1^{(j)} - \omega_0^{(j+1)}\Delta_j.$$

We can express these results concisely by the use of a vector notation. Write

$$_j p = [{}_j p_{12}, {}_j p_{13}], \qquad \nu_0^{(j)} = [\nu_0^{(j)}, \omega_0^{(j)}], \qquad \nu_1^{(j)} = [\nu_1^{(j)}, \omega_1^{(j)}]$$

and

$$D_j = \begin{bmatrix} {}_j\delta_{22}, & {}_j\delta_{23} \\ {}_j\delta_{32}, & {}_j\delta_{33} \end{bmatrix}.$$

Then we have

$$v_0^{(j+1)} = p_{j+1} + v_0^{(j)} D_{j+1} \quad \text{and} \quad v_1^{(j+1)} = v_1^{(j)} D_{j+1} - v_0^{(j+1)} \Delta_j. \tag{8}$$

By the use of (5), (4) and (8) we can express $\Delta\mu_2^{(j)}$, and hence $\mu_2^{(n)}$, in terms of the initial probabilities and the D_j, and so in terms of the initial probabilities and the elements of the P_j of the chain.

4. The Convergence of the Variance

We wish to show that the value of $\mu_2^{(n)}/n$ for the chain in which $P_n \to P$ tends, as $n \to \infty$, to that for the homogeneous chain with P as its matrix of transition probabilities. It is sufficient to show that the corresponding $\Delta\mu_2^{(n)}$ tend to coincidence, and this follows if we show that $v_1^{(n)}$, $v_0^{(n)}$ and Δ_n tend to coincidence with the corresponding $v_1^{(n)}$, $v_0^{(n)}$ and Δ_n for the homogeneous chain. We consider first $v_0^{(n)}$.

By the use of (8) we have that

$$v_0^{(n)} = p_n + p_{n-1}D_n + p_{n-2}D_{n-1}D_n + \ldots + p_0 D_1 D_2 \ldots D_n, \tag{9}$$

where $p_0 = [\beta, \gamma]$. If we define p and D similarly for the associated homogeneous chain we have likewise in that case

$$v_0^{(n)} = p + pD + pD^2 + \ldots + pD^{n-1} + p_0 D^n. \tag{10}$$

We wish to compare (9) and (10) and for this need the following lemma, which we state and prove with reference to the general case of a chain with k states. We use the notation $|B| < b$ to mean that each element of the matrix B does not exceed b in modulus, and $P^{[n]}$ to denote a product of n matrices P_j.

Lemma.—*If* $|P_j - P| < \epsilon$ *for all* j *then* $|P^{[n]} - P^n| < nk\epsilon$.

Proof.—The result clearly holds for $n = 1$. Write $P_j = P + E_j$, and assume that the result is true for $n = m$. Then, for some j,

$$\begin{aligned} |P^{[m+1]} - P^{m+1}| &= |P_j P^{[m]} - P^{m+1}| \\ &= |(P + E_j)P^{[m]} - P.P^m| \\ &= |P(P^{[m]} - P^m) + E_j P^{[m]}| \\ &< |P(P^{[m]} - P^m)| + |E_j P^{[m]}| \\ &< mk\epsilon + k\epsilon \end{aligned}$$

since P has non-negative elements and unit row sums. It follows by induction that the result is true for all values of n, since we certainly have from the inequality above that $| P^{[m+1]} - P^{m+1} | < (m+1)k\epsilon$.

We now consider the difference between the expressions as given by (9) and (10) for $v_0^{(n)}$ in the non-homogeneous and associated homogeneous cases. We shall once again treat the general case of a chain with k states; in this case expressions formally equivalent to those of (9) and (10) occur, the only difference being that the elements of each D_j, which is now of order $k-1$, arise by subtraction of the first row of the corresponding P_j from the remaining $k-1$ rows.

We can write the difference of the $v_0^{(n)}$ as

$$\begin{aligned}\{(p_n - p) + (p_{n-1}D_n - pD) + \ldots + (p_{n-j_1+1}D_{n-j_1+2} \ldots D_n - pD^{j_1-1})\} \\ + \{(p_{n-j_1}D_{n-j_1+1} \ldots D_n - pD^{j_1}) + \ldots + (p_0 D_1 D_2 \ldots D_n - p_0 D^n)\} \\ = A + B,\end{aligned}$$

say, there being j_1 pairs of terms in A and $n - j_1 + 1$ pairs of terms in B.

Consider B first. We suppose to begin with that P is itself positive so that we have $| P | > 2e$ for some $e > 0$. Then since $P_j \to P$ as $j \to \infty$ we have that $| P_j - P | < e$ for j sufficiently large, $j > j_0$ say, so that $| P_j | > e$ for $j > j_0$. Let $P^{[r]}$ and $D^{[r]}$ now denote the products of r matrices P_j, D_j respectively for all of which $j > j_0$, and let $d = 1 - e < 1$. We define the *range* ρ_j of a finite stochastic matrix $P_j = [{}_jp_{\alpha\beta}]$ by

$$\rho_j = \max_{\alpha, \alpha', \beta} \{{}_jp_{\alpha\beta} - {}_jp_{\alpha'\beta}\}.$$

It follows that $| D_j | < \rho_j$. Then if f_j denotes the least element of P_j we have, for $j > j_0$, that $1 - f_j < d$ so the range of $P^{[r]}$ does not exceed d^r (Mott 1957, p. 372) and hence $| D^{[r]} | < d^r$. Also, since $1 - 2e < d$, we have that $| D^r | < d^r$. Further, the range of a product $P_a P_b$ of finite stochastic matrices does not exceed the range of P_b, for each element of a given column of $P_a P_b$ is a weighted mean of the elements of the corresponding column of P_b so that the greatest element of a column of $P_a P_b$ can not exceed, nor the least element be less than, the greatest and least elements respectively of the corresponding column of P_b. Thus $| (D_{j_0-\alpha} \ldots D_{j_0})(D_{j_0+1} \ldots D_m) | < d^{m-j_0}$, where $0 \leq \alpha \leq j_0 - 1$ and $m \geq j_0 + 1$. Now take $n - j_0 + 1 > j_0$ and note the result that $| pD | < d$ and the analogous results for a product of D_j, and we have that

$$\begin{aligned}| B | &< 2\{d^{j_1} + d^{j_1+1} + \ldots + d^{n-j_0} + j_0 d^{n-j_0}\} \\ &< 2\{j_0 d^{j_1} + d^{j_1}/(1-d)\}.\end{aligned}$$

Thus, given any $\epsilon > 0$, we can choose j_1 such that $| B | < \frac{1}{2}\epsilon$.

The value of n is still arbitrary except that $n > j_0 + j_1 - 1$, and we now show that by a suitable choice of n we can ensure that likewise $|A| < \frac{1}{2}\epsilon$. To do this we use the lemma. Given any $\epsilon' > 0$ we have that $|P_j - P| < \epsilon'$ for j sufficiently large, $j > n_1$ say. If now $P^{[r]}$ and $D^{[r]}$ denote the products of r matrices P_j, D_j respectively for each of which $j > n_1$ we have from the lemma that $|P^{[r]} - P^r| < rk\epsilon'$ and hence that $|D^{[r]} - D^r| < 2rk\epsilon'$, so that

$$\begin{aligned} |p_j D^{[r]} - pD^r| &= |(p_j - p)D^{[r]} + p(D^{[r]} - D^r)| \\ &\leqslant |(p_j - p)D^{[r]}| + |p(D^{[r]} - D^r)| \\ &< k\epsilon' + 2rk\epsilon' \\ &< 3rk\epsilon'. \end{aligned}$$

Thus if we take $n - j_1 + 1 > n_1$ (as we can consistently with the condition on n above) we have that

$$\begin{aligned} |A| &< \epsilon' + 3k\epsilon' j_1(j_1 - 1)/2 \\ &< 3k\epsilon' j_1(j_1 + 1)/2. \end{aligned}$$

If we now choose $\epsilon' = \{3kj_1(j_1 + 1)\}^{-1}\epsilon$ we have that $|A| < \frac{1}{2}\epsilon$ and so $|A + B| < \epsilon$. It follows that, on the supposition that P is positive, the $v_0^{(n)}$ in the non-homogeneous and associated homogeneous cases tend to coincidence as $n \to \infty$.

In the general case of P positive-regular P itself may be not positive. However, P^n is positive for sufficiently large values of n and we have $|P^n| > 2e$ for some $e > 0$ if $n \geqslant n_2$ say. Also $|P_n - P| < e/kn_2$ for n sufficiently large, $n > n_3$ say, so that if $P^{[n]}$ now denotes the product of n matrices P for each of which $j > n_4 = \max\{n_2, n_3\}$ we have from the lemma that $|P^{[n_2]} - P^{n_2}| < e$ so that $|P^{[n_2]}| > e$. It follows for a product $D^{[rn_2]}$ of rn_2 matrices D_j for each of which $j < n_4$ that $|D^{[rn_2]}| < d^r$. We can now proceed to consider B much as in the preceding particular case; we treat groups of n_2 terms throughout and use the result that $|D^{[rn_2+a]}| \leqslant |D^{[rn_2]}|$. Thus we can take $j_1 = (r + 1)n_2$ and n such that $n - j_1 + 1 > n_4$ and have that

$$|B| < 2kn_2\{d + d^2 + \ldots\} + 2k(n_4 + n_2)d^{n_5},$$

where n_5 is the integral part of $(n - n_4)/n_2$. Thus

$$|B| < 2k(n_4 + n_2)d^{n_5} + 2kn_2 d/(1 - d).$$

Thus for j_1 and n sufficiently large we have $|B| < \frac{1}{2}\epsilon$ for any given $\epsilon > 0$. We can now show that by the choice of n sufficiently large we can ensure that $|A| < \frac{1}{2}\epsilon$; this we do by making the changes in the previous proof for A corresponding to those we have just made for B. We omit the details.

Thus the expressions (9) and (10) for $v_0^{(n)}$ in the non-homogeneous and associated homogeneous cases tend to coincidence as $n \to \infty$. It follows from (4) that the two corresponding expressions for Δ_n also tend to coincidence; but this we know already since $P^{(n)} \to P^n$.

Now consider the expression for $v_1^{(n)}$ as given by (8). For the moment let us write v_j for $v_1^{(j)}$ and u_j for $v_0^{(j)}\Delta_{j-1}$. With this notation (8) becomes

$$v_{j+1} = v_j D_{j+1} - u_j$$

so that

$$v_n = v_0 D_1 D_2 \ldots D_n - u_1 D_2 \ldots D_n - u_2 D_3 \ldots D_n - \ldots - u_{n-1} D_n - u_n.$$

We have a corresponding expression for the v_n of the associated homogeneous chain, and have to show that these two expressions tend to equality as $n \to \infty$. But since $D_1 \ldots D_n$ tends to zero as $n \to \infty$ and the u_n for the non-homogeneous chain is already known to tend to the u_n for the associated homogeneous chain we have essentially the same problem as that treated above for $v_0^{(n)}$, and the result follows in the same way.

Finally we have from (5), since $P_n \to P$ as $n \to \infty$, that the two expressions for $\Delta\mu_2^{(n)}$ tend to coincidence. It follows that the variances of the probability distributions of $x_n/n^{\frac{1}{2}}$ in the non-homogeneous and associated homogeneous cases tend to coincidence as $n \to \infty$.

The assumption that P is positive-regular ensures that this variance is non-zero (Fréchet 1938, p. 88); this is essential to the following proof of normality.

5. Asymptotic Relations between the Higher Moments

[5.1] We now prove (1) and (2) for the convergent non-homogeneous chain. For ease of printing we shall often omit a suffix b in reference to a Δb and to the elements of a p_b or of a D_b, and likewise omit the superfix b in reference to a $\mu_i^{(b)}$, $\nu_i^{(b)}$, $\omega_i^{(b)}$. The occurrence of these suffixes and superfixes has been illustrated throughout § 3, which also provides an example, for a particular case of moments of low order, of the methods of the present general case. The proof is by induction. (1) is certainly true for $m = 1$ and is an immediate consequence of

$$\Delta\mu_{2r}^{(n)} = \binom{2r}{2}\mu_{2r-2}^{(n)}.\Delta\mu_2^{(n)} + O(n^{r-2}) \qquad (11)$$

for $r = m$, provided that (11) is true for all $r < m - 1$. To prove (11) we return to the recurrence relation (3). We show below (in § 5.2) that if (11) is true for all $r < m - 1$ then the orders of $\nu_r^{(n)}$ and $\omega_r^{(n)}$ are not higher than that of $\mu_r^{(n)}$ for all $r < m - 1$; for the moment we assume this result and

so have from (3) that $\mu_{2m}^{(n+1)}$ is, apart from terms $O(n^{m-2})$, the coefficient of $(it)^{2m}/(2m)!$ in

$$\{1-\Delta(it)+\Delta^2(it)^2/2!+\ldots\}\{\mu_{2m-2}(it)^{2m-2}/(2m-2)!+[\mu_{2m-1}$$
$$+(2m-1)(p_{11}\mu_{2m-2}+\delta_{21}\nu_{2m-2}+\delta_{31}\omega_{2m-2})](it)^{2m-1}/(2m-1)!+[\mu_{2m}+2m(p_{11}\mu_{2m-1}$$
$$+\delta_{21}\nu_{2m-1}+\delta_{31}\omega_{2m-1})+m(2m-1)(p_{11}\mu_{2m-2}+\delta_{21}\nu_{2m-2}+\delta_{31}\omega_{2m-2})](it)^{2m}/(2m)!\}.$$

Then, using (4) in the form $\Delta-p_{11}=\delta_{21}\nu_0+\delta_{31}\omega_0$, we have that

$$\begin{aligned}\Delta\mu_{2m}&=2m[(\nu_{2m-1}-\nu_0\mu_{2m-1})\delta_{21}+(\omega_{2m-1}-\omega_0\mu_{2m-1})\delta_{31}]\\&\quad+m(2m-1)[\mu_{2m-2}p_{11}+\nu_{2m-2}\delta_{21}+\omega_{2m-2}\delta_{31}-\mu_{2m-2}\Delta^2]\\&\quad+2m(2m-1)[(\nu_{2m-2}-\nu_0\mu_{2m-2})\delta_{21}+(\omega_{2m-2}-\omega_0\mu_{2m-2})\delta_{31}]+O(n^{m-2}).\end{aligned}$$

On the introduction of $\Delta\mu_2$ by the use of (5) we find that the latter equation is equivalent to

$$\begin{aligned}&2m\{[(\nu_{2m-1}-\nu_0\mu_{2m-1})-(2m-1)\nu_1\mu_{2m-2}]\delta_{21}+[(\omega_{2m-1}-\omega_0\mu_{2m-1})\\&-(2m-1)\omega_1\mu_{2m-2}]\delta_{31}\}+m(2m-1)\{(\nu_{2m-2}-\nu_0\mu_{2m-2})\delta_{21}+(\nu_{2m-2}-\omega_0\mu_{2m-2})\delta_{31}\}\\&=O(n^{m-2}).\end{aligned}$$

This is so if

$$\nu_{2r}^{(n)}-\nu_0^{(n)}\mu_{2r}^{(n)}=O(n^{r-1})\quad\text{and}\quad\omega_{2r}^{(n)}-\omega_0^{(n)}\mu_{2r}^{(n)}=O(n^{r-1});\tag{12}$$

and

$$\left.\begin{aligned}(\nu_{2r-1}^{(n)}-\nu_0^{(n)}\mu_{2r-1}^{(n)})-(2r-1)\nu_1^{(n)}\mu_{2r-2}^{(n)}&=O(n^{r-2})\\(\omega_{2r-1}^{(n)}-\omega_0^{(n)}\mu_{2r-1}^{(n)})-(2r-1)\omega_1^{(n)}\mu_{2r-2}^{(n)}&=O(n^{r-2})\end{aligned}\right\}\tag{13}$$

for $r\leqslant m+1$.

[5.2] We have from (9), or more readily from (10) to which (9) tends as $n\to\infty$, that $\nu_0^{(n)}=O(1)$. To show this we have merely to note that, with the notation of § 4, $|D^n|<d^{n_3}$ for $n>n_4$ so that

$$|\nu_0^{(n)}|<n_4+n_2+n_2(d+d^2+\ldots)<n_4+n_2/(1-d)=O(1).$$

It follows from (8) in a similar way that $\nu_1^{(n)}=O(1)$ also.

These particular results provide a basis for an inductive proof of the general result that

$$\nu_{2m-2}^{(n)}=\omega_{2m-2}^{(n)}=\nu_{2m-1}^{(n)}=\omega_{2m-1}^{(n)}=O(n^{m-1})$$

if $\mu_{2r}^{(n)}=\mu_{2r+1}^{(n)}=O(n^r)$ for all $r<m-1$, that is, if (11) and (2) hold for all $r<m-1$. For from (3) $\nu_{2r}^{(n+1)}$ is then the coefficient of $(it)^{2r}/(2r)!$ in

$$\{1\}[\mu_{2r}(it)^{2r}/(2r)!,\quad\nu_{2r}(it)^{2r}/(2r)!,\quad\omega_{2r}(it)^{2r}/(2r)!][p_{12},\delta_{22},\delta_{32}]'+O(n^{r-1}),$$

and likewise for $\omega_{2r}^{(n+1)}$, $\nu_{2r+1}^{(n+1)}$ and $\omega_{2r+1}^{(n+1)}$. This is the result that we anticipated above in § 5.1.

[5.3] We now prove (12). We assume that (12) is true for $r \leqslant m-2$ and show that in consequence it is true for $r = m-1$. We have that

$$\nu_{2m-2}^{(n+1)} = p_{12}\mu_{2m-2} + \delta_{22}\nu_{2m-2} + \delta_{32}\omega_{2m-2} + O(n^{m-2})$$

so that, on elimination of p_{12} by the use of (7),

$$\nu_{2m-2}^{(n+1)} - \nu_0^{(n+1)}\mu_{2m-2}^{(n)} = \delta_{22}(\nu_{2m-2} - \nu_0\mu_{2m-2}) + \delta_{32}(\omega_{2m-2} - \omega_0\mu_{2m-2}) + O(n^{m-2}).$$

We can replace $\mu_{2m-2}^{(n)}$ by $\mu_{2m-2}^{(n+1)}$ since $\Delta\mu_{2m-2}^{(n)} = O(n^{m-2})$ by our assumption that (11) is true for $r \leqslant m-1$ and so have that

$$\nu_{2m-2}^{(n+1)} - \nu_0^{(n+1)}\mu_{2m-2}^{(n+1)} = \delta_{22}(\nu_{2m-2} - \nu_0\mu_{2m-2}) + \delta_{32}(\omega_{2m-2} - \omega_0\mu_{2m-2}) + O(n^{m-2}). \quad (14)$$

Similarly

$$\omega_{2m-2}^{(n+1)} - \omega_0^{(n+1)}\mu_{2m-2}^{(n+1)} = \delta_{23}(\nu_{2m-2} - \nu_0\mu_{2m-2}) + \delta_{33}(\omega_{2m-2} - \omega_0\mu_{2m-2}) + O(n^{m-2}). \quad (15)$$

Now write

$$\nu_{2m-2}^{(j)} - \nu_0^{(j)}\mu_{2m-2}^{(j)} = x_j, \qquad \omega_{2m-2}^{(j)} - \omega_0\mu_{2m-2}^{(j)} = y_j \quad \text{and} \quad \underline{x}_j = [x_j, y_j].$$

Then with D_j as before (14) and (15) can be combined in the vector equation

$$\underline{x}_{j+1} = \underline{x}_j D_{j+1} + \underline{a}_{j+1}, \quad (16)$$

where $\underline{a}_{j+1}$ is a vector whose elements are $O(j^{m-2})$. From (16) we have that

$$\underline{x}_n = \underline{a}_n + \underline{a}_{n-1}D_n + \underline{a}_{n-2}D_{n-1}D_n + \ldots + \underline{a}_0 D_1 D_2 \ldots D_n. \quad (17)$$

Thus (12) follows if

$$D_n + D_{n-1}D_n + \ldots + D_1 D_2 \ldots D_n = O(1),$$

and this is so by the arguments used earlier in § 5.2 and in § 4.

[5.4] We now prove (13). The proof is analogous to that of (12): we assume that (13) is true for $r \leqslant m-2$ and show that it is then true for $r = m-1$. We then have from the recurrence relation (3) that $\nu_{2m-1}^{(n+1)}$ is, apart from terms $O(n^{m-2})$, the coefficient of $(it)^{2m-1}/(2m-1)!$ in

$$\{1 - \Delta(it)\}\{(p_{12}\mu_{2m-2} + \delta_{22}\nu_{2m-2} + \delta_{32}\omega_{2m-2})(it)^{2m-2}/(2m-2)! + (p_{12}\mu_{2m-1} + \delta_{22}\nu_{2m-1} + \delta_{32}\omega_{2m-1})(it)^{2m-1}/(2m-1)!\}.$$

Then using (4) to eliminate p_{12} and also (12) with $r = m-1$ we have that

$$\nu_{2m-1}^{(n+1)} - \nu_0^{(n+1)}\mu_{2m-1}^{(n)} = \delta_{22}(\nu_{2m-1} - \nu_0\mu_{2m-1}) + \delta_{32}(\omega_{2m-1} - \omega_0\mu_{2m-1}) - (2m-1)\nu_0^{(n+1)}\mu_{2m-2}\cdot\Delta + O(n^{m-2}).$$

By (7) we can write this equation in the form

$$\nu_{2m-1}^{(n+1)} - \nu_0^{(n+1)}\mu_{2m-1}^{(n)} - (2m-1)\nu_1^{(n+1)}\mu_{2m-2}^{(n)} + O(n^{m-2}) = \delta_{22}[(\nu_{2m-1} - \nu_0\mu_{2m-1}) - (2m-1)\nu_1\mu_{2m-2}] + \delta_{32}[(\omega_{2m-1} - \omega_0\mu_{2m-1}) - (2m-1)\omega_1\mu_{2m-2}]. \quad (18)$$

We now bring the left-hand side of this equation to its natural form much as in the proof of (12) in § 5.3: we use the result of § 5.1 that $\Delta\mu_{2m-2}^{(n)} = O(n^{m-2})$ and the result of § 5.5 below that $\Delta\mu_{2m-1}^{(n)} = O(n^{m-2})$. Thus we can replace the left-hand side of (18) by

$$\nu_{2m-1}^{(n+1)} - \nu_0^{(n+1)}\mu_{2m-1}^{(n+1)} - (2m-1)\nu_1^{(n+1)}\mu_{2m-2}^{(n+1)} \tag{19}$$

without change of the order of the omitted terms. The corresponding equation for

$$\omega_{2m-1}^{(n+1)} - \omega_0^{(n+1)}\mu_{2m-1}^{(n+1)} - (2m-1)\omega_1^{(n+1)}\mu_{2m-2}^{(n+1)} \tag{20}$$

follows similarly. We can express the two recurrence relations for (19) and (20) by a single vector equation identical in form with (16) above, and then complete the proof exactly as for (12) in § 5.3.

[5.5] We have still to begin the process of induction but, apart from this, have given an inductive proof of (11) provided that $\Delta\mu_{2m-1}^{(n)} = O(n^{m-2})$. We now show that this is so provided that

$$\Delta\mu_{2r+1}^{(n)} = O(n^{r-1}) \tag{21}$$

for all $r \leqslant m-2$. We have from (3) that $\mu_{2m-1}^{(n+1)}$ is, apart from terms $O(n^{m-2})$, the coefficient of $(it)^{2m-1}/(2m-1)!$ in

$$\{1 - \Delta(it)\}\{\mu_{2m-2}(it)^{2m-2}/(2m-2)! + [\mu_{2m-1} + (2m-1)(\mu_{2m-2}p_{11} + \nu_{2m-2}\delta_{21} + \omega_{2m-2}\delta_{31})](it)^{2m-1}/(2m-1)!\}$$

if (1) and (2) are true for all $r \leqslant m-2$. On use of (4) we have that

$$\Delta\mu_{2m-1}^{(n)} = (2m-1)[(\nu_{2m-2} - \nu_0\mu_{2m-2})\delta_{21} + (\omega_{2m-2} - \omega_0\mu_{2m-2})\delta_{31}] + O(n^{m-2}).$$

We now use (12) for $r = m-1$, and the result follows. (2) is an immediate consequence of (21).

[5.6] To begin the process of induction we note that (11) is true for $r = 1$ (this is immediate), that (12) is true for $r = 1$ and that (13) is true for $r = 2$. But clearly (12) is true for $r = 1$ from (17), and likewise for (13) with $r = 2$. This completes the proof of (1) and (2).

6. The Proof for a Chain with k States

The convergence of the distribution of $x_n/n^{\frac{1}{2}}$ to normality follows at once from (1) and (2) (Fréchet and Shohat 1931). We have given the details of our proof for the particular case of $k = 3$, but the extension of this proof to general values of k presents a problem of notation only. We have given in § 4 the details of the convergence of the variance in the general case, and in § 5 we have corresponding changes: in (16), for example, $\underline{x}_j$ is now a $(k-1)$-dimensional vector and the D_j are of order $k-1$. The inductive argument is unchanged.

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IX.—The Propagation of Thermal Stresses in Thin Metallic Rods.*

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SYNOPSIS

If the temperature in an elastic rod is not uniform and if it varies with time, dynamic thermal stresses are set up in the rod. This paper is concerned with the calculation of the distribution of temperature and stress in an elastic rod when its ends are subjected to mechanical or thermal disturbances. Simple waves in an infinite rod are first discussed and then boundary value problems for semi-infinite rods and rods of finite length. The paper concludes with an account of an approximate method of solving the equations of thermoelasticity.

1. INTRODUCTION

IF the distribution of temperature in an elastic body is not uniform, dilatational changes take place which alter the distribution of stress in the body. On the other hand if dilatational waves are being propagated in an elastic body the local changes in volume will produce fluctuations of temperature in the body. Thermoelasticity, which is concerned with the study of the interplay of these two effects, has recently been the subject of several papers (Biot 1956; Lessen 1956 and 1957) although the basic equations of the subject have been known for quite some time (Duhamel 1837; Voigt 1910, Jeffreys 1930).

The present paper is concerned with what is probably the simplest physical system in which thermoelastic phenomena can occur—a thin metallic rod; in this case the basic equations assume their simplest forms. The solutions of problems relating to even this simple system are of physical interest and the methods used to obtain these solutions might be expected to have obvious generalizations in more complicated situations.

After a brief account (in §§ 2, 3, 4) of the basic equations of thermoelasticity with one space variable and of the system of units we employ, we give an account of the propagation of simple waves in an infinite rod whose cylindrical surface is impervious to heat. Exact and approximate expressions are derived for the phase velocity and the attenuation coefficients of these waves and numerical values for four metals (aluminium, copper,

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iron and lead) calculated over a wide range of the frequency of the wave. We introduce a frequency, ω^*, characteristic of the solid, defined by the equation $\omega^* = cE/k$, and a constant $\epsilon = \alpha^2 E T/\rho c$ where c, E, k, α, and ρ are respectively the specific heat, the Young's modulus, the conductivity, the coefficient of linear expansion and the density of the metal; T is the absolute temperature of the rod in a state of zero strain. It is shown that if the frequency of the wave is less than ω^* the phase velocity is $(1 + \frac{1}{2}\epsilon)(E/\rho)^{\frac{1}{2}}$ and the attenuation coefficient is $\frac{1}{4}\epsilon(2 - 5\epsilon)(E/\rho)^{\frac{1}{2}}(\omega^2/^*\omega^3)$; these results are the one-dimensional analogue of the results for the dilatational waves in a three-dimensional solid (Deresiewicz 1957; Chadwick and Sneddon 1958).

The remainder of the paper is concerned with boundary value problems associated with semi-infinite rods of finite length. In § 6 we consider semi-infinite rods whose free ends are subjected to simple periodic disturbances and in § 7 make use of the theory of Laplace transforms to solve similar problems with disturbances of a quite general nature. The propagation of thermoelastic waves in finite rods is discussed briefly in § 8. The paper concludes with an account of a method of deriving approximate solutions of the basic equations.

2. The Thermoelastic Equations

Under free thermal expansion a thin rod experiences a thermal strain in the direction of the rod of amount

$$\epsilon^{(1)} = \alpha\theta, \tag{2.1}$$

where θ denotes the temperature change from T, the absolute temperature of the rod in a state of zero stress and strain, and α denotes the coefficient of linear expansion of the solid. It is assumed that θ is so small that the thermal and elastic properties of the solid remain constant throughout the times in which we are interested. If we measure position along the rod by means of a co-ordinate x and denote the displacement of a point with co-ordinate x by u then the total strain at a typical point x is given by

$$\epsilon = \frac{\partial u}{\partial x}. \tag{2.2}$$

This total strain is made up of the thermal strain and the elastic strain $\epsilon^{(2)}$ which is given by the equation

$$\epsilon^{(2)} = \frac{\sigma}{E}, \tag{2.3}$$

where σ denotes the stress at the point x and E denotes the Young's modulus of the material of the rod. Substituting from equations (2.1),

(2.2) and (2.3) into the equation

$$\epsilon = \epsilon^{(1)} + \epsilon^{(2)} \tag{2.4}$$

we obtain the equation

$$\sigma = E\frac{\partial u}{\partial x} - \gamma\theta, \tag{2.5}$$

where

$$\gamma = \alpha E. \tag{2.6}$$

Equation (2.5) is the one-dimensional form of the *Duhamel-Neumann law* (Sokolnikoff 1956, p. 359).

The thermodynamic variables describing the state of the elastic rod are the strain ϵ and the absolute temperature $T+\theta$. It has been shown by Biot (1956) that the entropy per unit volume of an elastic rod is given by

$$s = c\rho \log_e\left(1+\frac{\theta}{T}\right) + \gamma\epsilon, \tag{2.7}$$

where c is the specific heat per unit mass at constant strain (assumed independent of variations in the temperature in the vicinity of the equilibrium temperature T), ρ is the density, and the additive constant involved in the definition of the entropy has been chosen so that the entropy is zero in the reference state. If θ is small in comparison with T we find that

$$s = \frac{c\rho\theta}{T} + \gamma\epsilon$$

so that the quantity of heat absorbed per unit volume of the solid in the course of small deformations and small variations in temperature is given by the formula

$$h = Ts = c\rho\theta + \gamma T\frac{\partial u}{\partial x}. \tag{2.8}$$

Now it is known from the theory of the linear conduction of heat in solids that when the surface of the rod is rendered impervious to heat, so that no radiation takes place, the equation for the variation of temperature takes the form

$$\frac{\partial h}{\partial t} = k\frac{\partial^2\theta}{\partial x^2} \tag{2.9}$$

(Carslaw and Jaeger 1947, p. 6) where k is the thermal conductivity of the material of the rod. If we substitute from equation (2.8) into equation (2.9) and introduce the diffusity

$$\kappa = \frac{k}{\rho c}, \tag{2.10}$$

we can write equation (2.9) in the form

$$\frac{\partial\theta}{\partial t}=\kappa\frac{\partial^2\theta}{\partial x^2}-\gamma'\frac{\partial^2 u}{\partial x\partial t}, \tag{2.11}$$

where $\gamma'=\gamma T/(\rho c)$.

Equation (2.11) resembles the customary equation for the linear flow of heat except for the term $\gamma'(\partial^2 u/\partial x\partial t)$. Duhamel (1837) included a similar term in his considerations but only because he postulated that the rate of change of dilatation of an elastic body would have a linear effect on the rate of change of temperature. Alternative proofs of equation (2.11) have been given by Voigt (1910) and Jeffreys (1930) and more recently by Lessen (1953, 1956, 1957).

To complete the set of basic equations we have the equation of motion

$$\frac{\partial\sigma}{\partial x}+\rho F=\rho\frac{\partial^2 u}{\partial t^2}, \tag{2.12}$$

where F denotes the body force per unit mass in the direction of the rod.

The set of equations (2.5), (2.11) and (2.12) are sufficient, when taken with the appropriate boundary conditions, to determine the variation of temperature, stress and displacement along the rod when the body force F is prescribed. Wiener (1957) has proved that solutions of these equations are unique when θ and u are specified on the ends of the rod and the initial distributions of θ, u and $\partial u/\partial t$ are prescribed; his proof can readily be extended to cover more general boundary value problems.

3. Dimensionless Form of the Equations

It is convenient to write the basic thermoelastic equations in dimensionless form. If we take a typical length l as our unit of length, a time τ as our unit of time, the reference temperature T as our unit of temperature, and the Young's modulus E as unit of stress, we find that equations (2.12), (2.5) and (2.11) respectively assume the forms

$$\frac{\partial\sigma}{\partial x}+X=a\frac{\partial^2 u}{\partial t^2}, \tag{3.1}$$

$$\sigma=\frac{\partial u}{\partial x}-b\theta, \tag{3.2}$$

$$\frac{\partial^2\theta}{\partial x^2}=f\frac{\partial\theta}{\partial t}+g\frac{\partial^2 u}{\partial x\partial t}, \tag{3.3}$$

where

$$a=\left(\frac{l}{U\tau}\right)^2, \qquad b=\alpha T, \qquad f=\frac{\rho c l^2}{k\tau}, \qquad g=\frac{\alpha E l^2}{k\tau} \tag{3.4}$$

with $U=(E/\rho)^{\frac{1}{2}}$, the velocity of elastic waves in the rod. It should be noted that b is independent of the choice of l and τ as also is

$$\epsilon=\frac{bg}{f}=\frac{\alpha^2 ET}{\rho c}. \tag{3.5}$$

If we are interested in low frequencies it is convenient to take l to be 1 cm. and τ to be 1 sec. The values of a, b, f, g and ϵ for four metals are shown in Table I; here we have taken $l=1$ cm., $\tau=1$ sec. and $T=293°\text{K}=20°$ C.

TABLE I

	Aluminium	Copper	Iron	Lead
a . .	$4{\cdot}029\times10^{-12}$	$7{\cdot}539\times10^{-12}$	$5{\cdot}984\times10^{-12}$	$7{\cdot}063\times10^{-11}$
b . .	$7{\cdot}62\times10^{-3}$	$4{\cdot}98\times10^{-3}$	$1{\cdot}026\times10^{-2}$	$9{\cdot}67\times10^{-3}$
f . .	1·168	0·899	5·208	4·152
g . .	0·860	0·479	3·536	1·470
$\epsilon=bg/f$.	$5{\cdot}61\times10^{-3}$	$2{\cdot}65\times10^{-3}$	$6{\cdot}97\times10^{-3}$	$3{\cdot}42\times10^{-3}$

For high frequencies it is more appropriate to take

$$\tau=\frac{1}{\omega^*}, \qquad l=\frac{U}{\omega^*}, \tag{3.6}$$

where U denotes the elastic wave velocity and ω^* denotes the frequency

$$\omega^*=\frac{\rho c}{k}U^2. \tag{3.7}$$

For this choice of units we have

$$a=1, \qquad b=\alpha T, \qquad f=1, \qquad g=\frac{\alpha E}{\rho c}. \tag{3.8}$$

TABLE II

	Aluminium	Copper	Iron	Lead
U(km./sec.) .	5·09	3·52	5·21	1·19
ω^*(sec.$^{-1}$) .	$3{\cdot}03\times10^{11}$	$1{\cdot}11\times10^{11}$	$1{\cdot}43\times10^{12}$	$5{\cdot}88\times10^{11}$
l(cm.) . .	$1{\cdot}68\times10^{-6}$	$3{\cdot}16\times10^{-6}$	$3{\cdot}64\times10^{-7}$	$2{\cdot}02\times10^{-7}$
b . . .	$7{\cdot}62\times10^{-3}$	$4{\cdot}98\times10^{-3}$	$1{\cdot}03\times10^{-2}$	$9{\cdot}67\times10^{-3}$
g . . .	0·736	0·533	0·679	0·354
ω_c(sec.$^{-1}$) .	$7{\cdot}90\times10^{13}$	$6{\cdot}10\times10^{13}$	$8{\cdot}94\times10^{13}$	$2{\cdot}05\times10^{13}$

The values of ω^* and l for aluminium, copper, iron and lead are given in Table II. Again it is assumed that $T=20°$ C. The range of frequencies actually obtainable in a solid is limited above by the cut-off frequency ω_c

of the Debye spectrum. For longitudinal vibrations in a bar

$$\omega_c = 2\pi U\left(\frac{3\rho}{4\pi M}\right)^{\frac{1}{3}} \tag{3.9}$$

(Brillouin 1938, p. 324), M being the mass of an atom of the metal forming the bar. The values of ω_c are included in Table II.

In practical problems still another system of units may be employed. For instance, if we are considering the propagation of stress in a rod of length 1 metre, then it is obviously desirable to take $l = 10^2$ cm. and we may choose

$$\tau = 10^2/U. \tag{3.10}$$

With this choice of units we find that we may write the equations in the forms

$$\frac{\partial \sigma}{\partial x} = \frac{\partial^2 u}{\partial t^2}, \tag{3.11}$$

$$\sigma = \frac{\partial u}{\partial x} - b\theta, \tag{3.12}$$

$$\xi \frac{\partial^2 \theta}{\partial x^2} = \frac{\partial \theta}{\partial t} + \frac{g}{f}\frac{\partial^2 u}{\partial x \partial t}, \tag{3.13}$$

where

$$\xi = \frac{1}{f} = \frac{k}{\rho c U} \times 10^{-2}, \qquad \frac{g}{f} = \frac{aE}{\rho c}, \qquad b = aT \tag{3.14}$$

all the physical quantities being measured in c.g.s. units. For the metals we have been considering we get the values of ξ and τ given in Table III;

TABLE III

	Aluminium	Copper	Iron	Lead
τ (sec.) . .	$1{\cdot}97 \times 10^{-4}$	$2{\cdot}84 \times 10^{-4}$	$1{\cdot}92 \times 10^{-4}$	$8{\cdot}40 \times 10^{-4}$
ξ . . .	$1{\cdot}68 \times 10^{-8}$	$3{\cdot}16 \times 10^{-8}$	$3{\cdot}68 \times 10^{-9}$	$2{\cdot}02 \times 10^{-8}$

in this system of units g/f has the same value as g has in Table II, and b has the same value as in Tables I and II (on the assumption that $T = 20°$ C.).

4. RADIATION FROM A ROD

Equation (2.11) governs the flow of heat in the rod when the surface of the rod is rendered impervious to heat. The equation has to be modified when radiation takes place into a medium at constant temperature, T say.

The conduction equation (2.9) must then be replaced by the equation

$$A\frac{\partial h}{\partial t}=Ak\frac{\partial^2\theta}{\partial x^2}-pH\theta, \tag{4.1}$$

where A is the same cross-sectional area of the rod, p is the perimeter of a cross-section and H is the emissivity of the surface. Because of this equation (2.11) must be replaced by

$$\frac{\partial\theta}{\partial t}=\kappa\frac{\partial^2\theta}{\partial x^2}-\gamma'\frac{\partial^2 u}{\partial x\partial t}-\gamma''\theta, \tag{4.2}$$

where

$$\gamma''=\frac{Hp}{c\rho A}. \tag{4.3}$$

This means that in turn equation (3.3) must be replaced by

$$\frac{\partial^2\theta}{\partial x^2}=f\frac{\partial\theta}{\partial t}+g\frac{\partial^2 u}{\partial x\partial t}+j\theta, \tag{4.4}$$

where f and g have the same values as before and

$$j=\frac{Hpl^2}{Ak}. \tag{4.5}$$

5. Simple Waves in an Infinite Rod

We shall consider first of all the propagation of waves in a rod whose surface is impervious to heat. If we put $X=0$ in equation (3.1) and put each of the quantities u, σ, θ proportional to $e^{i\omega t}$, then equations (3.1), (3.2) and (3.3) become

$$D\sigma=-a\omega^2 u, \tag{5.1}$$

$$\sigma=Du-b\theta, \tag{5.2}$$

$$D^2\theta=i\omega f\theta+i\omega gDu, \tag{5.3}$$

where $D=d/dx$. It follows that each of the independent variables satisfies the equation

$$\begin{vmatrix} a\omega^2 & D & 0 \\ D & -1 & -b \\ i\omega gD & 0 & -(D^2-i\omega f) \end{vmatrix}\phi=0,$$

which may be written in the form

$$(D^2-\mu_1^2)(D^2-\mu_2^2)\phi=0, \tag{5.4}$$

where μ_1^2, μ_2^2 are the roots of the quadratic equation

$$(\mu^2+a\omega^2)(\mu^2-i\omega f)-i\omega bg\mu^2=0.$$

If we write

$$\Delta^2=[a\omega^2-i(f+bg)\omega]^2+4iaf\omega^3, \tag{5.5}$$

then we have

$$\mu_1^2=\tfrac{1}{2}[-a\omega^2+i\omega(f+bg)+\Delta], \tag{5.6}$$

$$\mu_2^2=\tfrac{1}{2}[-a\omega^2+i\omega(f+bg)-\Delta] \tag{5.7}$$

and the linearly independent solutions of equation (5.4) are the set of equations (3.1), (3.2) and (3.3) therefore have solutions of the form

$$\begin{pmatrix}u\\ \sigma\\ \theta\end{pmatrix}=\begin{pmatrix}\mu_1\\ \sigma_1\\ \theta_1\end{pmatrix}e^{\mu_1 x+i\omega t}+\begin{pmatrix}u_2\\ \sigma_2\\ \theta_2\end{pmatrix}e^{-\mu_1 x+i\omega t}+\begin{pmatrix}u_3\\ \sigma_3\\ \theta_3\end{pmatrix}e^{\mu_2 x+i\omega t}+\begin{pmatrix}u_4\\ \sigma_4\\ \theta_4\end{pmatrix}e^{-\mu_2 x+i\omega t}$$

where the u_i, σ_i, θ_i $(i=1, 2, 3, 4)$ are constants. Of these twelve constants only four can be chosen arbitrarily; the remaining eight must be chosen in such a way that the equations (5.1), (5.2) and (5.3) are satisfied identically.

It is obvious from an examination of equations (5.6) and (5.7) that μ_2 is the root which corresponds to the longitudinal elastic wave. Now the phase velocity of the wave $\phi=\phi_0\exp(-\mu_2 x+i\omega t)$ is given by the equation

$$V=\frac{\omega}{\Im(\mu_2)} \tag{5.8}$$

and the attenuation constant is

$$q=\Re(\mu_2). \tag{5.9}$$

In problems of this kind the most appropriate units are those of Table II. Putting $a=f=1$, $bg=\alpha^2ET/\rho c=\epsilon$ in equations (5.6) and (5.7) we then find that

$$\mu_1^2=\tfrac{1}{2}[-\omega^2+i(1+\epsilon)\omega+\Delta], \qquad \mu_2^2=\tfrac{1}{2}[-\omega^2+i(1+\epsilon)\omega-\Delta], \tag{5.10}$$

where

$$\Delta^2=[\omega^2-i(1+\epsilon)]^2+4i\omega^3. \tag{5.11}$$

Remembering that, in these units, the unit of velocity is U and the unit of

length is U/ω^* we find from equations (5.8) and (5.9) that

$$V=\frac{\omega}{\Im(\mu_2)}U, \qquad q=\Re(\mu_2)\frac{U}{\omega^*}, \tag{5.12}$$

where μ_2 is given by the second of equations (5.10).

The algebraic expressions for the roots μ_1, μ_2 would be very cumbersome in the general case but it is a simple matter to approximate to them if ω is either very small or very large in comparison with unity (*i.e.* in comparison with ω^*). If $\omega \ll 1$ in this system of units then it is easily shown that μ_1 and μ_2 take the approximate values $\mu_1^{(0)}$, $\mu_2^{(0)}$ defined by the equations

$$\mu_1^{(0)}=\pm(\tfrac{1}{2}\omega)^{\frac{1}{2}}(1+\tfrac{1}{2}\epsilon)(1+i), \qquad \mu_2^{(0)}=\pm\{\tfrac{1}{2}\epsilon(1-\tfrac{5}{2}\epsilon)+(1-\tfrac{1}{2}\epsilon)i\omega\}, \tag{5.13}$$

where it has been assumed that ϵ also is small. Similarly if $\omega \gg 1$ in this system it can be shown that μ_1 and μ_2 take the approximate values $\mu_1^{(\infty)}$, $\mu_2^{(\infty)}$ where

$$\mu_1^{(\infty)}=\pm(\tfrac{1}{2}\omega)^{\frac{1}{2}}\left\{\left(1-\frac{\epsilon}{2\omega}\right)+i\left(1+\frac{\epsilon}{2\omega}\right)\right\}, \qquad \mu_2^{(\infty)}=\pm(\tfrac{1}{2}\epsilon+i\omega). \tag{5.14}$$

It follows from equations (5.13), (5.14) and (5.12) that in "ordinary" units

$$q=\begin{cases}(1-\frac{5}{2}\epsilon)q_\infty(\omega/\omega^*)^2, & \omega \ll \omega^*,\\ q_\infty, & \omega \gg \omega^*,\end{cases} \tag{5.15}$$

where

$$q_\infty=\tfrac{1}{2}\epsilon\frac{U}{\omega^*} \tag{5.16}$$

and from equation (5.12) that for $\omega \ll \omega^*$

$$V=(1+\tfrac{1}{2}\epsilon)U, \tag{5.17}$$

while for $\omega \gg \omega^*$, $V=U$.

The attenuation constant q is therefore an increasing function of the frequency ω of the waves varying like ω^2 for low frequencies and approaching the value q_∞ asymptotically as $\omega \to \infty$. Equation (5.17) shows that for low frequencies, *i.e.* for frequencies less than 10^{10} sec.$^{-1}$, the phase velocity of the longitudinal elastic waves is $(1+\frac{1}{2}\epsilon)$ times that of the elastic wave in a medium not exhibiting a thermal effect. This result can be interpreted in a different way. If measurements of the velocity of longitudinal waves in rods were used to determine the Young's modulus of the material of the rods then the "dynamical" value E_d of the Young's modulus would be related to the statical value E through the equation

$$E_d=(1+\epsilon)E.$$

From the values given in Table I (p. 125) we see that in the case of aluminium the dynamical value will be 0·6 per cent higher than the statical value, for copper it will be 0·3 per cent higher, for iron 0·7 per cent higher and for lead 0·3 per cent higher. The effect of the thermal properties of the bar on these values is therefore rather slight.

TABLE IV

ω/ω^*	V/U			
	Aluminium	Copper	Iron	Lead
10^{-2} . .	1·0028	1·0013	1·0035	1·0017
10^{-1} . .	1·0028	1·0013	1·0034	1·0017
1 . .	1·0014	1·0007	1·0018	1·0009
10 . .	1·0000	1·0000	1·0000	1·0000
10^2 . .	1·0000	1·0000	1·0000	1·0000
$1+\frac{1}{2}\varepsilon$ . .	1·0028	1·0013	1·0035	1·0017

In order to test the validity of these arguments a series of calculations based on equations (5.6), (5.7), (5.8) and (5.9) was carried out for the four metals listed in Table I for a range of frequencies extending from $10^{-11}\omega^*$ to $10^3\omega^*$. A programme for carrying out these calculations on DEUCE, the high-speed electronic computer installed at the National Physical Laboratory, Teddington, was devised by Mr J. H. Wilkinson of the Mathematics Division of that Laboratory. Some of the results obtained from this programme are shown in Tables IV and V. It will be observed

TABLE V

ω/ω^*	q/q_∞			
	Aluminium	Copper	Iron	Lead
10^{-2} . .	$0{\cdot}9846\times10^{-4}$	$0{\cdot}9917\times10^{-4}$	$0{\cdot}9815\times10^{-4}$	$0{\cdot}9910\times10^{-4}$
10^{-1} . .	$0{\cdot}9863\times10^{-2}$	$0{\cdot}9837\times10^{-2}$	$0{\cdot}9729\times10^{-2}$	$0{\cdot}9828\times10^{-2}$
1 . .	0·4991	0·4996	0·4988	0·5000
10 . .	0·9899	0·9901	0·9896	0·9910
10^2 . .	0·9995	0·9999	0·9993	1·0000
$1-\frac{5}{2}\varepsilon$ . .	0·9860	0·9933	0·9826	0·9914

from Table IV that if $\omega/\omega^*=10^{-2}$ then $V=(1+\frac{1}{2}\epsilon)U$; further calculations on DEUCE (not reported in full here) confirm that over the range $10^{-11}<\omega/\omega^*<10^{-2}$ the phase velocity of longitudinal elastic waves in the rod has this constant value. When $\omega=\omega^*$ the phase velocity of these waves is almost exactly $(1+\frac{1}{4}\epsilon)U$ and when $\omega>10\omega^*$ it falls to the value U, independent of ϵ. Table V shows the values of q/q_∞ for the same range of frequencies; for higher values of ω it was found that $q=q_\infty$ while for $\omega<10^{-2}\omega^*$ it was found that q was given accurately by equation (5.15).

It is obvious from these results that there is a sharp change in the values of V and q in the vicinity of the frequency ω^*. It would seem therefore that the frequency ω^*, which was introduced for the purely mathematical reason that its choice put the basic equations in a simple form, has in fact a definite physical significance.

6. Semi-infinite Rods: Simple Boundary Conditions

We shall consider now the state of stress and the distribution of temperature in the semi-infinite rod $x \geqslant 0$ when the end $x=0$ is disturbed by the application of stresses or temperatures which have a time dependence of the sinusoidal form $e^{i\omega t}$ where ω is real.

If we assume that the amplitudes of the thermal and elastic waves do not increase indefinitely as $x \to \infty$ then we must take values of μ_1, μ_2 with negative real parts. It is readily verified that if μ_1 and μ_2 are given by equations (5.10), negative real parts being selected, then the expressions

$$u(x, t) = \frac{\mu_1 b}{\mu_1^2 + a\omega^2} C_1 e^{\mu_1 x + i\omega t} + \frac{\mu_2 b}{\mu_2^2 + a\omega^2} C_2 e^{\mu_2 x + i\omega t}, \tag{6.1}$$

$$\sigma(x, t) = -ba\omega^2 \left\{ \frac{C_1}{\mu_1^2 + a\omega^2} e^{\mu_1 x + i\omega t} + \frac{C_2}{\mu_2^2 + a\omega^2} e^{\mu_2 x + i\omega t} \right\}, \tag{6.2}$$

$$\theta(x, t) = C_1 e^{\mu_1 x + i\omega t} + C_2 e^{\mu_2 x + i\omega t}, \tag{6.3}$$

satisfy the basic equations (3.1), (3.2) and (3.3) with $X=0$. These solutions lead to the boundary values

$$u(0, t) = \left\{ \frac{\mu_1 b C_1}{\mu_1^2 + a\omega^2} + \frac{\mu_2 b C_2}{\mu_2^2 + a\omega^2} \right\} e^{i\omega t}, \tag{6.4}$$

$$\sigma(0, t) = -ba\omega^2 \left\{ \frac{C_1}{\mu_1^2 + a\omega^2} + \frac{C_2}{\mu_2^2 + a\omega^2} \right\} e^{i\omega t}, \tag{6.5}$$

$$\theta(0, t) = (C_1 + C_2) e^{i\omega t}, \tag{6.6}$$

$$\frac{\partial \theta(0, t)}{\partial x} = (\mu_1 C_1 + \mu_2 C_2) e^{i\omega t}. \tag{6.7}$$

Case (i):

$$\theta(0, t) = \Theta e^{i\omega t}, \qquad \sigma(0, t) = 0.$$

For this set of boundary conditions we must choose C_1 and C_2 so that

$$C_1 + C_2 = \Theta, \qquad (\mu_2^2 + a\omega^2) C_1 + (\mu_1^2 + a\omega^2) C_2 = 0,$$

from which it follows that

$$C_1 = -\frac{\mu_1^2 + a\omega^2}{\mu_2^2 - \mu_1^2}\Theta, \qquad C_2 = \frac{\mu_2^2 + a\omega^2}{\mu_2^2 - \mu_1^2}\Theta,$$

so that the desired solution is

$$u(x, t) = -\frac{b\Theta}{\mu_2^2 - \mu_1^2}(\mu_1 e^{\mu_1 x + i\omega t} - \mu_2 e^{\mu_2 x + i\omega t}). \tag{6.8}$$

$$\sigma(x, t) = \frac{ab\omega^2\Theta}{\mu_2^2 - \mu_1^2}(e^{\mu_1 x + i\omega t} - e^{\mu_2 x + i\omega t}), \tag{6.9}$$

$$\theta(x, t) = -\frac{\Theta}{\mu_2^2 - \mu_1^2}[(\mu_1^2 + a\omega^2)e^{\mu_1 x + i\omega t} - (\mu_2^2 + a\omega^2)e^{\mu_2 x + i\omega t}]. \tag{6.10}$$

In particular

$$u(0, t) = \frac{b\Theta}{\mu_1 + \mu_2}e^{i\omega t}. \tag{6.11}$$

Case (ii):

$$\theta(0, t) = \Theta e^{i\omega t}, \qquad u(0, t) = 0.$$

In a similar way we can show that the solution of this boundary value problem is

$$u(x, t) = -\frac{\mu_1\mu_2\Theta}{(\mu_2 - \mu_1)(\mu_1\mu_2 - a\omega^2)}(e^{\mu_1 x + i\omega t} - e^{\mu_2 x + i\omega t}), \tag{6.12}$$

$$\sigma(x, t) = \frac{ba\omega^2\Theta}{(\mu_2 - \mu_1)(\mu_1\mu_2 - a\omega^2)}(\mu_2 e^{\mu_1 x + i\omega t} - \mu_1 e^{\mu_2 x + i\omega t}), \tag{6.13}$$

$$\theta(x, t) = -\{\mu_2(\mu_1^2 + a\omega^2)e^{\mu_1 x + i\omega t} - \mu_1(\mu_2^2 + a\omega^2)e^{\mu_2 x + i\omega t}\}\frac{\Theta}{(\mu_2 - \mu_1)(\mu_1\mu_2 - a\omega^2)}. \tag{6.14}$$

In particular

$$\sigma(0, t) = \frac{ba\omega^2\Theta}{\mu_1\mu_2 - a\omega^2}e^{i\omega t}.$$

Since $\mu_1^2\mu_2^2 = -iaf\omega^3$ it follows that in this case

$$\sigma(0, t) = b\Theta\left(\frac{a\omega}{2f}\right)^{\frac{1}{2}}(1 + i)e^{i\omega t}. \tag{6.15}$$

Case (iii):

$$\sigma(0, t) = \Pi e^{i\omega t}, \qquad \partial\theta(0, t)/\partial x = 0.$$

In this instance the constants C_1 and C_2 are determined by the equations

$$\mu_1 C_1 + \mu_2 C_2 = 0, \qquad \frac{C_1}{\mu_1^2 + a\omega^2} + \frac{C_2}{\mu_2^2 + a\omega^2} = -\frac{\Pi}{ba\omega^2}$$

so that

$$\frac{C_1}{\mu_2} = -\frac{C_2}{\mu_1} = \frac{\Pi(\mu_1^2 + a\omega^2)(\mu_2^2 + a\omega^2)}{ba\omega(\mu_1 - \mu_2)(\mu_1^2 + \mu_1\mu_2 + \mu_2^2 + a\omega^2)}. \tag{6.16}$$

The solution of the boundary value problem is given by equations (6.1), (6.2) and (6.3) with C_1 and C_2 given by the equations (6.16). In particular we have the boundary expressions

$$\theta(0, t) = -\Pi \frac{(\mu_1^2 + a\omega^2)(\mu_2^2 + a\omega^2)}{ba\omega^2(\mu_1^2 + \mu_1\mu_2 + \mu_2^2 + a\omega^2)} e^{i\omega t}, \tag{6.17}$$

$$u(0, t) = -\Pi \frac{\mu_1\mu_2(\mu_1 + \mu_2)}{a\omega^2(\mu_1^2 + \mu_1\mu_2 + \mu_2^2 + a\omega^2)} e^{i\omega t}. \tag{6.18}$$

Case (iv):

$$\sigma(0, t) = \Pi e^{i\omega t}, \qquad \theta(0, t) = 0.$$

The solution of this problem is

$$u(x, t) = \frac{\Pi}{a\omega^2(\mu_1^2 - \mu_2^2)}\{\mu_1(\mu_2^2 + a\omega^2)e^{\mu_1 x + i\omega t} - \mu_2(\mu_1^2 + a\omega^2)e^{\mu_2 x + i\omega t}\}, \tag{6.19}$$

$$\sigma(x, t) = -\frac{\Pi}{(\mu_1^2 - \mu_2^2)}\{(\mu_2^2 + a\omega^2)e^{\mu_1 x + i\omega t} - (\mu_1^2 + a\omega^2)e^{\mu_2 x + i\omega t}\}, \tag{6.20}$$

$$\theta(x, t) = \frac{\Pi(\mu_1^2 + a\omega^2)(\mu_2^2 + a\omega^2)}{ba\omega^2(\mu_1^2 - \mu_2^2)}(e^{\mu_1 x + i\omega t} - e^{\mu_2 x + i\omega t}), \tag{6.21}$$

whence

$$u(0, t) = -\frac{(\mu_1\mu_2 - a\omega^2)\Pi e^{i\omega t}}{a\omega^2(\mu_1 + \mu_2)}. \tag{6.22}$$

Case (v):

$$u(0, t) = Ae^{i\omega t}, \qquad \partial\theta/\partial x = 0.$$

The solution of this boundary value problem is

$$u(x, t) = \frac{A}{\mu_2^2 - \mu_1^2}\{(\mu_2^2 + a\omega^2)e^{\mu_1 x + i\omega t} - (\mu_1^2 + a\omega^2)e^{\mu_2 x + i\omega t}\}, \tag{6.23}$$

$$\sigma(x, t) = -\frac{a\omega^2 A}{\mu_2^2 - \mu_1^2}\left\{\frac{\mu_2^2 + a\omega^2}{\mu_1} e^{\mu_1 x + i\omega t} - \frac{\mu_1^2 + a\omega^2}{\mu_2} e^{\mu_2 x + i\omega t}\right\}, \tag{6.24}$$

$$\theta(x, t) = \frac{A(\mu_1^2 + a\omega^2)(\mu_2^2 + a\omega^2)}{b(\mu_2^2 - \mu_1^2)}(\mu_1^{-1} e^{\mu_1 x + i\omega t} - \mu_2^{-1} e^{\mu_2 x + i\omega t}) \tag{6.25}$$

from which we have

$$\sigma(0, t) = -\frac{a\omega^2 A(\mu_1^2 + \mu_1\mu_2 + \mu_2^2 + a\omega^2)}{\mu_1\mu_2(\mu_1 + \mu_2)} e^{i\omega t}, \tag{6.26}$$

$$\theta(0, t) = \frac{A(\mu_1^2 + a\omega^2)(\mu_2^2 + a\omega^2)}{b\mu_1\mu_2(\mu_1 + \mu_2)} e^{i\omega t}. \tag{6.27}$$

Case (vi):

$$u(0, t) = Ae^{i\omega t}, \qquad \theta(0, t) = 0.$$

In this case we have

$$u(x, t) = \frac{A}{(\mu_2 - \mu_1)(\mu_1\mu_2 - a\omega^2)}\{\mu_1(\mu_2^2 + a\omega^2)e^{\mu_1 x + i\omega t} - \mu_2(\mu_1^2 + a\omega^2)e^{\mu_2 x + i\omega t}\} \tag{6.28}$$

$$\sigma(x, t) = -\frac{a\omega^2 A}{(\mu_2 - \mu_1)(\mu_1\mu_2 - a\omega_2)}\{(\mu_2^2 + a\omega^2)e^{\mu_1 x + i\omega t} - (\mu_1^2 + a\omega^2)e^{\mu_2 x + i\omega t}\}, \tag{6.29}$$

$$\theta(x, t) = \frac{A(\mu_1^2 + a\omega^2)(\mu_2^2 + a\omega^2)}{b(\mu_2 - \mu_1)(\mu_1\mu_2 - a\omega^2)}(e^{\mu_1 x + i\omega t} - e^{\mu_2 x + i\omega t}), \tag{6.30}$$

and hence

$$\sigma(0, t) = -\frac{a\omega^2(\mu_1 + \mu_2)Ae^{i\omega t}}{\mu_1\mu_2 - a\omega^2}. \tag{6.31}$$

The reciprocal nature of the pair of equations (6.31) and (6.22) will be observed; the pair (6.18) and (6.26) are similarly related.

7. Semi-infinite Rods: Solution by the Laplace Transform

In this section we shall consider the conditions in the semi-infinite rod $x > 0$ when the end $x = 0$ is subjected to certain stresses and thermal conditions. It is assumed that at $t = 0$ the stress and displacement are zero as is the deviation θ of the temperature from the reference temperature.

If we assume that there are no body forces present so that we may put $X = 0$ in equation (3.1) then it is readily seen that the set of equations (3.1), (3.2) and (3.3) is equivalent to the set of ordinary differential equations

$$D\bar{\sigma} = as^2\bar{u}, \tag{7.1}$$

$$\bar{\sigma} = D\bar{u} - b\bar{\theta}, \tag{7.2}$$

$$D^2\bar{\theta} = sf\bar{\theta} + gsD\bar{u} \tag{7.3}$$

($D=d/dx$), for the Laplace transforms

$$(\bar{u}, \bar{\sigma}, \bar{\theta})=\int_0^\infty (u, \sigma, \theta)e^{-st}dt \tag{7.4}$$

of u, σ and θ. It is readily seen that each of these quantities satisfies the fourth order ordinary differential equation

$$(D^2-\kappa_1^2)(D^2-\kappa_2^2)\bar{\phi}=0, \tag{7.5}$$

where

$$\kappa_1^2+\kappa_2^2=as^2+(f+bg)s, \qquad \kappa_1^2\kappa_2^2=afs^3. \tag{7.6}$$

On the assumption that σ, u and θ all tend to zero (or at worst remain finite) as $x\to\infty$ we take solutions of equations (7.1), (7.2) and (7.3) of the form

$$\bar{u}(x, s)=-b\left\{\frac{\kappa_1 C_1 e^{-\kappa_1 x}}{\kappa_1^2-as^2}+\frac{\kappa_2 C_2 e^{-\kappa_2 x}}{\kappa_2^2-as^2}\right\}, \tag{7.7}$$

$$\bar{\sigma}(x, s)=abs^2\left\{\frac{C_1 e^{-\kappa_1 x}}{\kappa_1^2-as^2}+\frac{C_2 e^{-\kappa_2 x}}{\kappa_2^2-as^2}\right\}, \tag{7.8}$$

$$\bar{\theta}(x, s)=C_1 e^{-\kappa_1 x}+C_2 e^{-\kappa_2 x},$$

where it is assumed that $\Re(\kappa_1)$ and $\Re(\kappa_2)$ are both positive, and C_1 and C_2 are determined from the conditions at $x=0$. To facilitate the calculation of C_1 and C_2 from the boundary conditions we have the relations

$$\bar{u}(0, s)=-b\left\{\frac{\kappa_1 C_1}{\kappa_1^2-as^2}+\frac{\kappa_2 C_2}{\kappa_2^2-as^2}\right\}, \tag{7.9}$$

$$\bar{\sigma}(0, s)=abs^2\left\{\frac{C_1}{\kappa_1^2-as^2}+\frac{C_2}{\kappa_2^2-as^2}\right\}, \tag{7.10}$$

$$\bar{\theta}(0, s)=C_1+C_2, \tag{7.11}$$

$$D\bar{\theta}(0, s)=-\kappa_1 C_1-\kappa_2 C_2. \tag{7.12}$$

It is obvious from this set of equations that if the boundary values of any two of the four quantities $\bar{u}$, $\bar{\sigma}$, $\bar{\theta}$, $D\bar{\theta}$ are prescribed those of the remaining two are uniquely determined.

For instance, suppose that the stress is prescribed to have the value $\Pi(t)$ on the free end and it is assumed that there is no flux of heat across this end. Then $\sigma(0, t)=\Pi(t)$ and $\partial\theta(0, t)/\partial x=0$. From these conditions it follows immediately that $D\bar{\theta}(0, s)=0$ and that $\bar{\sigma}(0, s)=\bar{\Pi}(s)$, where $\bar{\Pi}(s)$ is the Laplace transform of the prescribed function $\Pi(t)$. Substituting these values in equations (7.10) and (7.12) and solving for C_1 and C_2 we find that

$$\frac{C_1}{-\kappa_2}=\frac{C_2}{\kappa_1}=\frac{g\bar{\Pi}(s)}{(\kappa_2-\kappa_1)[(f+bg)+a^{\frac{1}{2}}f^{\frac{1}{2}}s^{\frac{1}{2}}]}.$$

Substituting these values in equation (7.11) we obtain the relation

$$\bar{\theta}(0, s) = -\frac{g\bar{\Pi}(s)}{(f+bg)+a^{\frac{1}{2}}f^{\frac{1}{2}}s^{\frac{1}{2}}}.$$

If we choose $1/\omega^*$ as our unit of time and U/ω^* as our unit of length then $a=f=1$ and $bg=\epsilon$ so that we have

$$\bar{\theta}(0, s) = -\frac{g\Pi(s)}{s^{\frac{1}{2}}+\alpha}, \qquad \alpha = 1+\epsilon. \tag{7.13}$$

Now if we write

$$F(\alpha, t) = \pi^{-\frac{1}{2}}t^{-\frac{1}{2}} - \alpha e^{\alpha^2 t}\operatorname{Erfc}(\alpha t^{\frac{1}{2}}) \tag{7.14}$$

we find that

$$\bar{F}(\alpha, s) = \frac{1}{s^{\frac{1}{2}}+\alpha}$$

(Erdélyi *et al.* 1954, vol. 1, p. 233) so that making use of the Faltung theorem for Laplace transforms (Sneddon 1951, p. 31) we have the equation

$$\theta(0, t) = -g\int_0^t \Pi(t')F(1+\epsilon, t-t')dt' \tag{7.15}$$

by means of which to determine the variation of temperature at the free end ($x=0$) of the rod.

For example, if in conventional units

$$\Pi(t) = \begin{cases} -\mu p_0, & 0 < t < t_1 \\ 0, & t > t_1, \end{cases}$$

so that in our present units we have

$$\Pi(t) = \begin{cases} -p_0, & 0 < t < t_1\omega^* \\ 0, & t > t_1\omega^* \end{cases}$$

and it can be shown that

$$\theta(0, t) = \begin{cases} \dfrac{p_0 g}{\alpha}[1 - e^{\alpha^2 t}\operatorname{Erfc}(\alpha t^{\frac{1}{2}})], & 0 < t < t_1\omega^*, \\ \dfrac{p_0 g}{\alpha}[e^{\alpha^2(t-t_1\omega^*)}\operatorname{Erfc}\{\alpha(t-t_1\omega^*)^{\frac{1}{2}}\} - e^{\alpha^2 t}\operatorname{Erfc}(\alpha t^{\frac{1}{2}})], & t > t_1\omega^*. \end{cases}$$

Returning to conventional units we have

$$\theta(0, t) = \begin{cases} p_0\theta_m\{1 - e^{\gamma^2 t}\operatorname{Erfc}(\gamma t^{\frac{1}{2}})\}, & 0 < t < t_1, \\ p_0\theta_m\{e^{\gamma^2(t-t_1)}\operatorname{Erfc}[\gamma(t-t_1)^{\frac{1}{2}}] - e^{\gamma^2 t}\operatorname{Erfc}(\gamma t^{\frac{1}{2}})\}, & t > t_1, \end{cases} \tag{7.16}$$

where

$$\theta_m = \frac{gT}{1+\epsilon}, \qquad \gamma^2 = (1+\epsilon)^2\omega^*. \tag{7.17}$$

Values of θ_m and γ for the metals we have considered previously are given in Table VI for a reference temperature of 20° C.

TABLE VI

	Aluminium	Copper	Iron	Lead
θ_m(°K.) . . .	214·4	155·6	197·5	103·4
γ(sec.$^{-\frac{1}{2}}$) . . .	$5{\cdot}53 \times 10^5$	$3{\cdot}34 \times 10^5$	$1{\cdot}20 \times 10^6$	$7{\cdot}69 \times 10^5$
$\pi^{-\frac{1}{2}}\theta_m/\gamma$(°K. sec.$^{\frac{1}{2}}$) .	$2{\cdot}19 \times 10^{-4}$	$2{\cdot}63 \times 10^{-4}$	$9{\cdot}26 \times 10^{-5}$	$7{\cdot}58 \times 10^{-5}$

Since γ is very large we may use the asymptotic expansion

$$e^{x^2}\ \mathrm{Erfc}\,(x) \sim \frac{1}{\sqrt{\pi x}}\left(1 - \frac{1}{2x^2}\right) \tag{7.18}$$

to obtain the formulæ

$$\theta(0, t) = \begin{cases} p_0\theta_m\{1 - \gamma^{-1}(\pi t)^{-\frac{1}{2}}\} & 0 < t < t_1 \\ p_0\theta_m, & t = t_1 \\ \dfrac{p_0\theta_m}{\gamma\sqrt{\pi}}\{(t-t_1)^{-\frac{1}{2}} - t^{-\frac{1}{2}}\} & t > t_1. \end{cases} \tag{7.19}$$

Furthermore if $t \gg t_1$ we have the approximate formula

$$\theta(0, t) = \frac{p_0\theta_m}{\gamma(\pi t_1)^{\frac{1}{2}}}\left(\frac{t_1}{t}\right)^{\frac{3}{2}}. \tag{7.20}$$

8. Finite Rods

Similar methods may be employed to determine the state of stress and the distribution of temperature in the finite rod $0 \leqslant x \leqslant l$. There is a variety of thermoelastic problems associated with such a finite rod, the solution of each depending on the boundary conditions imposed.

In the first instance consider the vibrations possible in the rod when the ends $x = 0$ and $x = l$ are subjected to the conditions

$$\sigma = 0, \qquad \frac{\partial\theta}{\partial x} = 0. \tag{8.1}$$

It is readily verified that if μ_1 and μ_2 are given by equations (5.10), then the expression

$$\sigma = [\sigma_1 \sinh \mu_1 x + \sigma_2 \sinh \mu_2 x + \sigma_3(\cosh \mu_1 x - \cosh \mu_2 x)]e^{i\omega t}, \tag{8.2}$$

in which σ_1, σ_2, σ_3 are constants satisfies the equation (5.4) and the condition that $\sigma = 0$ when $x = 0$. From the relation

$$ab\omega^2\theta = -(D^2 + a\omega^2)\sigma$$

we deduce that the corresponding expression for θ is given by the equation

$$\theta = -\frac{1}{ab\omega^2}\{\sigma_1(\mu_1^2 + a\omega^2)\sinh \mu_1 x + \sigma_2(\mu_2^2 + a\omega^2)\sinh \mu_2 x + \sigma_3[(\mu_1^2 + a\omega^2)\cosh \mu_1 x - (\mu_2^2 + a\omega^2)\cosh \mu_2 x]\}e^{i\omega t}. \tag{8.3}$$

It follows immediately that the expressions (8.2) and (8.3) satisfy the boundary conditions (8.1) at $x = 0$ and $x = l$ provided that

$$\mu_1(\mu_1^2 + a\omega^2)\sigma_1 + \mu_2(\mu_2^2 + a\omega^2)\sigma_2 = 0,$$

$$\mu_1(\mu_1^2 + a\omega^2)\sigma_1 \cosh \mu_1 l + \mu_2(\mu_2^2 + a\omega^2)\sigma_2 \cosh \mu_2 l + [\mu_1(\mu_1^2 + a\omega^2)\sinh \mu_1 l - \mu_2(\mu_2^2 + a\omega^2)\sinh \mu_2 l]\sigma_3 = 0,$$

$$\sigma_1 \sinh \mu_1 l + \sigma_2 \sinh \mu_2 l + \sigma_3(\cosh \mu_1 l - \cosh \mu_2 l) = 0.$$

Eliminating the constants σ_1, σ_2, σ_3 from these equations we see that solutions of the type (8.2) and (8.3) are possible if ω is a root of the transcendental equation

$$\begin{vmatrix} \mu_1(\mu_1^2 + a\omega^2) & \mu_2(\mu_2^2 + a\omega^2) & 0 \\ \mu_1(\mu_1^2 + a\omega^2)\cosh \mu_1 l & \mu_2(\mu_2^2 + a\omega^2)\cosh \mu_2 l & \mu_1(\mu_1^2 + a\omega^2)\sinh \mu_1 l - \mu_2(\mu_2^2 + a\omega^2)\sinh \mu_2 l \\ \sinh \mu_1 l & \sinh \mu_2 l & \cosh \mu_1 l - \cosh \mu_2 l \end{vmatrix} = 0,$$

where μ_1 and μ_2 are defined in terms of ω by the equations (5.10). Expanding the determinant on the left-hand side of this equation we find that the frequency equation becomes

$$2\mu_1\mu_2(\mu_1^2 + a\omega^2)(\mu_2^2 + a\omega^2)(\cosh \mu_1 l \cosh \mu_2 l - 1) = [\mu_1^2(\mu_1^2 + a\omega^2)^2 + \mu_2^2(\mu_2^2 + a\omega^2)^2]\sinh \mu_1 l \sinh \mu_2 l. \tag{8.4}$$

The method of the Laplace transform can also be applied to problems concerning rods of finite length l. Suppose, for instance, that we have the following boundary conditions for the stress $\sigma(x, t)$ and the temperature variation $\theta(x, t)$:

$$\sigma(0, t) = \Pi(t), \qquad \frac{\partial\theta(0, t)}{\partial x} = 0$$
$$\sigma(l, t) = 0, \qquad \frac{\partial\theta(l, t)}{\partial x} = 0 \tag{8.5}$$

then we may take a solution of equation (7.5) in the form

$$\bar{\sigma}(x, s) = A \sinh \kappa_1(l - x) + B \sinh \kappa_2(l - x) + C \cosh \kappa_1(l - x) + D \cosh \kappa_2(l - x),$$

where κ_1, κ_2 are defined by the equations (7.6). The corresponding expression for the Laplace transform of the temperature variation is

$$-abs^2\bar{\theta}(x, s) \quad (\kappa_1^2 - as^2)[A \sinh \kappa_1(l-x) + C \cosh \kappa_1(l-x)] + (\kappa_2^2 - as^2)[B \sinh \kappa_2(l-x) + D \cosh \kappa_2(l-x)].$$

If these forms are to satisfy the boundary conditions (8.5) we must choose A, B, C, D so that they satisfy the equations

$$A \sinh \kappa_1 l + B \sinh \kappa_2 l + C \cosh \kappa_1 l + D \cosh \kappa_2 l = \Pi(s),$$

$$C + D = 0,$$

$$\kappa_1(\kappa_1^2 - as^2)(A \cosh \kappa_1 l + C \sinh \kappa_1 l) + \kappa_2(\kappa_2^2 - as^2)(B \cosh \kappa_2 l + D \sinh \kappa_2 l) = 0,$$

$$\kappa_1(\kappa_1^2 - as^2)A + \kappa_2(\kappa_2^2 - as^2)B = 0.$$

Solving these equations we find that

$$A = -\kappa_2(\kappa_2^2 - as^2)\phi, \qquad B = -\kappa_1(\kappa_1^2 - as^2)\phi, \qquad C = \psi, \qquad D = -\psi,$$

where

$$\phi = [\kappa_1(\kappa_1^2 - as^2) \sinh \kappa_1 l - \kappa_2(\kappa_2^2 - as^2) \sinh \kappa_2 l]\frac{\Pi(s)}{\Delta(s)},$$

$$\psi = -[\kappa_1\kappa_2(\kappa_1^2 - as^2)(\kappa_2^2 - as^2)(\cosh \kappa_1 l - \cosh \kappa_2 l)]\frac{\Pi(s)}{\Delta(s)}$$

with

$$\Delta(s) = 2\kappa_1\kappa_2(\kappa_1^2 - as^2)(\kappa_2^2 - as^2)(\cosh \kappa_1 l \cosh \kappa_2 l - 1) - [\kappa_1^2(\kappa_1^2 - as^2)^2 + \kappa_2^2(\kappa_2^2 - as^2)^2] \sinh \kappa_1 l \sinh \kappa_2 l. \tag{8.6}$$

The relation of the function $\Delta(s)$ defined by equation (8.6) to the frequency equation (8.4) is obvious.

9. Approximate Solutions

In this section we shall consider a method of obtaining approximate solutions to the complete set of equations (3.1) to (3.3).

In some cases it is a simple matter to find a set of functions $(\sigma_0, u_0, \theta_0)$ satisfying the boundary conditions of the problem and the "unlinked" equations

$$\frac{\partial \sigma_0}{\partial x} = a\frac{\partial^2 u_0}{\partial t^2}, \tag{9.1}$$

$$\sigma_0 = \frac{\partial u_0}{\partial x}, \tag{9.2}$$

$$\frac{\partial^2 \theta_0}{\partial x^2} = f\frac{\partial \theta_0}{\partial t}. \tag{9.3}$$

From these solutions we may determine an approximate solution of the set (3.1) to (3.3). Let

$$(u, \sigma, \theta) = \sum_{r=0}^{\infty} (u_r, \sigma_r, \theta_r) g^r, \tag{9.4}$$

where $(u_0, \sigma_0, \theta_0)$ satisfy equations (9.1) to (9.3); then, substituting in equations (3.1) to (3.3) we find that $(u_r, \sigma_r, \theta_r)$ satisfy the equations

$$\left(\frac{\partial^2}{\partial x^2} - a\frac{\partial^2}{\partial t^2}\right) u_r = \left(\frac{b}{g}\right)\frac{\partial \theta_{r-1}}{\partial x}, \tag{9.5}$$

$$\left(\frac{\partial^2}{\partial x^2} - f\frac{\partial}{\partial t}\right)\theta_r = \frac{\partial u_{r-1}}{\partial x}, \tag{9.6}$$

$$\sigma_r = \frac{\partial u_r}{\partial x} - \left(\frac{b}{g}\right)\theta_{r-1} \tag{9.7}$$

($r \geqslant 1$), and the relevant physical quantities vanish on the boundary of the rod.

As an example of this method, suppose that we wish to find the approximation solution of the problem considered in case (*i*) of § 6. In this case the approximate solutions of equations (9.1), (9.2) and (9.3) would be

$$\theta_0 = \Theta e^{i\omega t - (1+i)(\frac{1}{2}f\omega)^{\frac{1}{2}}x} \qquad u_0 = \sigma_0 = 0.$$

We therefore have

$$\left(\frac{\partial^2}{\partial x^2} - a\frac{\partial^2}{\partial t^2}\right) u_1 = -(1+i)(\tfrac{1}{2}f\omega)^{\frac{1}{2}}\left(\frac{b}{g}\right)\Theta e^{i\omega t - (1+i)(\frac{1}{2}f\omega)^{\frac{1}{2}}x},$$

which has solution

$$u_1 = Ae^{i\omega(t - a^{\frac{1}{2}}x)} - \frac{(b\Theta/g)(\frac{1}{2}f\omega)^{\frac{1}{2}}(1+i)e^{i\omega t - (1+i)(\frac{1}{2}f\omega)^{\frac{1}{2}}x}}{a\omega^2 + if\omega}.$$

Inserting this expression in equation (8.7) we find that

$$\sigma_1(x, t) = -ia^{\frac{1}{2}}\omega Ae^{i\omega(t - a^{\frac{1}{2}}x)} - \frac{(b\Theta/g)a\omega^2 e^{i\omega t - (1+i)(\frac{1}{2}f\omega)^{\frac{1}{2}}x}}{a\omega^2 + if\omega}$$

and the boundary conditions demand that $\sigma_1(0, t) = 0$ so that we have

$$A = \frac{ia^{\frac{1}{2}}\omega(b\Theta/g)}{a\omega^2 + if\omega}.$$

We therefore obtain the approximate solution

$$\tilde{u}(x, t) = gu_1(x, t)$$

$$= \frac{ib a^{\frac{1}{2}}\omega\Theta}{a\omega^2 + if\omega}e^{i\omega(t - a^{\frac{1}{2}}x)} - \frac{b(\frac{1}{2}f\omega)^{\frac{1}{2}}(1+i)\Theta}{a\omega^2 + if\omega}e^{-(1+i)(\frac{1}{2}f\omega)^{\frac{1}{2}}x + i\omega t}. \tag{9.8}$$

For this solution we have the boundary value

$$\tilde{u}(0, t) = -\left(\frac{2}{f\omega}\right)^{\frac{1}{2}} b\Theta \frac{1 - i[1 + (2a\omega/f)^{\frac{1}{2}}]}{1 + [1 + (2a\omega/f)^{\frac{1}{2}}]} e^{i\omega t}. \tag{9.9}$$

Comparing this solution with the exact solution (6.11) we find that

$$\chi(\omega) = \frac{|\tilde{u}(0, t)|}{|u(0, t)|} = \left(\frac{2}{f\omega}\right)^{\frac{1}{2}} \frac{|\mu_1 + \mu_2|}{[1 + \{1 + (2a\omega/f)^{\frac{1}{2}}\}^2]^{\frac{1}{2}}}. \tag{9.10}$$

To test the validity of this approximation the values of $\chi(\omega)$ for aluminium were examined over a wide range of frequencies. The results of this numerical investigation are shown in Table VII. It will be observed

TABLE VII

ω/ω^*	10^{-10}	10^{-8}	10^{-6}	10^{-4}	10^{-2}	1	10^{2}
$\chi(\omega)$	1·0028	1·0028	1·0028	1·0028	1·0024	1·0006	1·0000

that over a very wide frequency range the function $\chi(\omega)$ does not differ from unity by more than 0·0028 showing the approximate solution obtained by this crude method is accurate to within about one quarter of one per cent.

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X.—The Dynamic Stresses Produced in Elastic Bodies by Uneven Heating.

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Synopsis

The presence of a non-uniform distribution of temperature in an elastic solid gives rise to an additional term in the generalized Hooke's Law connecting the stress and strain tensors and to a term involving the time rate of change of the dilatation in the equation governing the conduction of heat in the solid. The present paper is concerned with the effects produced by these additional terms in two simple situations. In the first, the elastic solid is regarded as being of infinite extent and the distribution of temperature in the solid is produced by heat sources whose strength may vary with time. In the second, the solid is supposed to be semi-infinite and to be deformed by prescribed variations in the temperature of the bounding plane and by heat sources within itself.

CONTENTS

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I. The Theory of Thermoelastic Disturbances

1. *Introduction*

If there is a non-uniform distribution of temperature in an elastic solid then there are corresponding changes in volume elements throughout the solid. These changes in the strain distribution throughout the solid have two important consequences. First of all thermal strains are set up in the solid as a result of the expansions (or contractions) accompanying changes in temperature and these in turn affect the distribution of stress in the body. Furthermore some of the mechanical energy expended in producing local changes of volume in an elastic solid is converted into heat which is absorbed by the solid itself. Thermoelasticity is concerned with the analysis of the effect these interlinked phenomena have upon the distribution of stress and temperature in any given problem.

The study of thermoelastic phenomena seems to have been begun over a century ago by Duhamel (1837) but the first attempt at a rigorous derivation of the basic equations came later with the work of Voigt (1910) and Jeffreys (1930). Interest in the fundamentals of the subject has recently been revived by the publications of papers by Biot (1956) and Lessen (1956, 1957). Solutions of these equations have recently been considered by Deresiewicz (1957), Chadwick and Sneddon (1958), and Sneddon (1958).

The present paper is concerned with the derivation of general solutions of the equations of thermoelasticity in two simple situations—the first when the stresses are produced in an infinite solid by a given distribution of heat sources (Part II) and the second when the stresses are produced in a semi-infinite solid by imposing a prescribed distribution of temperature on the plane boundary of the solid (Part III). Throughout it is assumed that the displacements and strains of the solid are infinitesimal so that the classical theory of elasticity is applicable and that the fluctuations in the temperature within the solid are limited to a range within which the thermal and elastic constants of the material are independent of the temperature.

The first sections (§§ 2, 3) give a brief account of the fundamental equations of thermoelasticity and of the systems of units which may be employed to cast them into a convenient dimensionless form.

In § 4 we derive the general solution of the thermoelastic equations for an infinite solid in which are distributed heat sources of prescribed strength; the method of solution, based on the use of four-dimensional Fourier transforms, is similar to that employed in a recent paper on the equations of dynamical elasticity (Eason, Fulton and Sneddon 1956). From this general solution we derive the solution appropriate to steady state problems

(§ 5) and two-dimensional problems (§ 6). In § 7 we use a mixed Fourier–Hankel transform to derive the general solution of problems in which there is axial symmetry. In many problems (the quasi-static problems) it is possible to take one of the constants of the theory to be zero; the solutions appropriate to this assumption are discussed in § 8. We conclude our discussion of the infinite solid by considering (in § 9) the solutions of a number of special problems. In one of these (§ 9 (i)) it is possible to derive a formula by means of which we can estimate the closeness of the quasi-static approximation.

Part III follows the same general pattern. In § 10 we derive the general solution of the thermoelastic equations for a semi-infinite solid which is deformed by the application to its plane boundary of a prescribed distribution of temperature, and from it deduce the solutions appropriate to steady-state problems (§ 11) and to two-dimensional problems (§ 12). The paper ends with a brief discussion of a special problem—the determination of the effect produced by applying a periodic line source of temperature to the free surface.

2. *The Thermoelastic Equations*

Under free thermal expansion an isotropic body experiences a strain whose components $\gamma_{ij}^{(1)}$ referred to a set of orthogonal cartesian axes $O(x_1, x_2, x_3)$ are specified by the equation

$$\gamma_{ij}^{(1)} = a\theta\delta_{ij} \tag{2.1}$$

in which θ denotes the temperature changes from T, the temperature of the solid in a state of zero stress and strain, and a denotes the coefficient of linear expansion of the solid. It is assumed that θ is sufficiently small for the thermal properties of the solid to remain constant throughout the times in which we are interested. In terms of the components, u_i, of the displacement vector, the total strain in the solid is given by the equation

$$\gamma_{ij} = \tfrac{1}{2}(u_{i,j} + u_{j,i}), \tag{2.2}$$

where $u_{i,j}$ denotes the partial derivative $\partial u_i/\partial x_j$. This total strain is made up of the thermal strain and the elastic strain whose components $\gamma_{ij}^{(2)}$ are specified by the equation (Sokolnikoff 1956, p. 66)

$$\gamma_{ij}^{(2)} = \frac{\tau_{ij}}{2\mu} - \frac{\lambda\Phi\delta_{ij}}{2\mu(3\lambda + 2\mu)}, \tag{2.3}$$

where τ_{ij} are the components of the stress tensor,

$$\Phi = \tau_{ii} \tag{2.4}$$

is the sum of the principal stresses, λ and μ are Lamé's elastic constants for the body. Substituting from equations (2.1) to (2.3) into the equation

$$\gamma_{ij} = \gamma_{ij}^{(1)} + \gamma_{ij}^{(2)}$$

we obtain the tensor equation

$$\gamma_{ij} = \frac{\tau_{ij}}{2\mu} - \left\{ \frac{\lambda\Phi}{2\mu(3\lambda + 2\mu)} - \alpha\theta \right\} \delta_{ij}. \tag{2.5}$$

Solving this tensor equation for the components of the stress tensor we find that

$$\tau_{ij} = (\lambda\Delta - \gamma\theta)\delta_{ij} + 2\mu\gamma_{ij}, \tag{2.6}$$

where

$$\Delta = \gamma_{ii} = u_{i,i} \tag{2.7}$$

denotes the dilatation in the solid and

$$\gamma = \alpha(3\lambda + 2\mu). \tag{2.8}$$

The physical relationship expressed by the tensor equation (2.6) is called the Duhamel–Neumann Law (Duhamel 1837; Neumann 1885).

The thermodynamic variables describing the state of the elastic solid are the strain components (2.2) and the absolute temperature $T + \theta$, and it can be shown (Jeffreys 1930; Biot 1956) that the entropy s per unit volume of the solid is given by the equation

$$s = c\rho \log\left(1 + \frac{\theta}{T}\right) + \gamma\Delta, \tag{2.9}$$

where the additive constant, involved in the definition of the entropy, has been chosen so that the entropy is zero in the reference state. In this equation ρ is the density of the solid, c is the specific heat per unit mass at constant strain (assumed independent of temperature in the vicinity of the equilibrium temperature T), and γ is defined by equation (2.8). If θ is small in comparison with T we find that equation (2.9) gives the simple equation

$$s = \frac{\rho c\theta}{T} + \gamma\Delta \tag{2.10}$$

for the entropy per unit volume, so that the quantity of heat absorbed by unit volume of the solid in the course of small deformations and small variations in temperature is given by the formula

$$h = Ts = \rho c\theta + \gamma T\Delta. \tag{2.11}$$

Now it is known from the theory of the conduction of heat in solids (Carslaw and Jaeger 1947, p. 6) that the variation of temperature within an isotropic solid is governed by the equation

$$\frac{\partial h}{\partial t} = k\nabla^2\theta + q, \tag{2.12}$$

where k is the heat conductivity of the solid, and q is the quantity of heat per unit volume generated in the solid. Substituting from equation (2.11) into equation (2.12) we find that the temperature variation θ and the dilatation Δ are linked through the equation

$$\rho c\frac{\partial \theta}{\partial t} + \gamma T\frac{\partial \Delta}{\partial t} = k\nabla^2\theta + q. \tag{2.13}$$

If we introduce the diffusity

$$\kappa = \frac{k}{\rho c},$$

we can write this equation in the form

$$\frac{\partial \theta}{\partial t} = \kappa\nabla^2\theta + Q - \gamma'\frac{\partial \Delta}{\partial t}, \tag{2.14}$$

where $Q = q/(\rho c)$ and $\gamma' = \gamma T/(\rho c)$.

To complete the set of basic equations we have the equations of motion in the form (Sokolnikoff 1956, p. 370)

$$\tau_{ij,j} + \rho F_i = \rho\ddot{u}_i, \tag{2.15}$$

where (F_1, F_2, F_3) denotes the body force at the point (x_1, x_2, x_3) and $\ddot{u}_i$ denotes the i-th component, $\partial^2 u_i/\partial t^2$, of the acceleration of an infinitesimal element centred at the same point. It is of course assumed that the temperature changes involved are so small that the value of the elastic constants remain unaltered throughout the solid.

The set of sixteen equations symbolized by equations (2.2), (2.6), (2.14) and (2.15) is sufficient, when taken with the appropriate boundary conditions, to determine the temperature variation and the components of stress and displacement when the heat sources are prescribed, Weiner (1957) has proved that solutions of these equations with $Q = 0$, $F_i = 0$ $(i = 1, 2, 3)$ are unique when the initial values of θ, u_i and $\partial u_i/\partial t$ are all zero and θ and u_i are specified on the boundary of the solid being considered, but his proof can readily be extended to cover more general boundary conditions.

3. *Dimensionless Form of the Equations*

It is convenient to write the basic set of thermoelastic equations in dimensionless form (Sneddon and Berry 1958, p. 124). If we take a typical length l as our unit of length, a time τ as our unit of time, the reference temperature T as the unit of temperature, and the rigidity modulus μ as unit of stress we find that the equations (2.15), (2.6) and (2.14) respectively take the dimensionless forms

$$\tau_{pq,\,q} + X_p = a\ddot{u}_p, \tag{3.1}$$

$$\tau_{pq} = [(\beta^2 - 2)\Delta - b\theta]\delta_{pq} + 2\gamma_{pq}, \tag{3.2}$$

$$\nabla^2\theta + \Theta = f\frac{\partial\theta}{\partial t} + g\frac{\partial\Delta}{\partial t}, \tag{3.3}$$

where β is the ratio $(2 + \lambda/\mu)^{\frac{1}{2}}$,

$$X_p = \frac{l\rho}{\mu}F_p, \qquad \Theta = \frac{Ql^2}{kT} \tag{3.4}$$

define new source functions and

$$a = \left(\frac{l}{v_s\tau}\right)^2, \qquad b = \frac{\gamma T}{\mu}, \qquad f = \frac{l^2}{\kappa\tau}, \qquad g = \frac{\gamma l^2}{k\tau}, \tag{3.5}$$

v_s being the velocity of shear waves in the solid, *i.e.* $(\mu/\rho)^{\frac{1}{2}}$ (Bullen 1947, p. 21).

In certain problems in which there is axial symmetry it is desirable to employ cylindrical polar co-ordinates r, z and ϕ instead of cartesian co-ordinates. In the case of axial symmetry we may take the z axis to coincide with the axis of symmetry; the displacement vector then has physical components u and w in the r and z directions respectively and zero component normal to these directions. There are four non-vanishing physical components of stress which may be denoted, in the usual von Kármán notation, by σ_r, σ_ϕ, σ_z, τ_{rz} and which are related to u and w by the dimensionless equations

$$(\sigma_r, \sigma_\phi, \sigma_z) = [(\beta^2 - 2)\Delta - b\theta] + 2\left(\frac{\partial u}{\partial r}, \frac{u}{r}, \frac{\partial w}{\partial z}\right) \tag{3.6}$$

and

$$\tau_{rz} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial r}, \tag{3.7}$$

where the dilatation Δ takes the form

$$\Delta = \frac{\partial u}{\partial r} + \frac{u}{r} + \frac{\partial w}{\partial z} \tag{3.8}$$

in this system of co-ordinates. When transformed to cylindrical co-ordinates the equations of motion (3.1) assume, in the symmetrical case, the forms

$$\frac{\partial \sigma_r}{\partial r}+\frac{\partial \tau_{rz}}{\partial z}+\frac{\sigma_r-\sigma_\phi}{r}+F_r=a\ddot{u}, \tag{3.9}$$

$$\frac{\partial \tau_{rz}}{\partial r}+\frac{\partial \sigma_z}{\partial z}+\frac{\tau_{rz}}{r}+F_z=a\ddot{w} \tag{3.10}$$

and the diffusion equation (3.3) may be written as

$$\frac{\partial^2\theta}{\partial r^2}+\frac{1}{r}\frac{\partial \theta}{\partial r}+\frac{\partial^2\theta}{\partial z^2}+\Theta=f\frac{\partial \theta}{\partial t}+g\frac{\partial \Delta}{\partial t}. \tag{3.11}$$

The values of the constants a, b, f and g occurring in these equations will depend upon the choice of the basic units of time and length. In some problems it may be convenient to choose l to be 1 cm. and τ to be 1 sec. The values of a, b, f and g for four common metals and this choice of l and τ are shown in Table I; here we have taken T to be 293°K. For

TABLE I

Values of the constants a, b, f and g for four common metals with $l=1$ cm., $\tau=1$ sec. and $T=293$°K.

	Aluminium	Copper	Iron	Lead
a . .	$1{\cdot}034\times10^{-11}$	$2{\cdot}166\times10^{-11}$	$1{\cdot}532\times10^{-11}$	$2{\cdot}034\times10^{-10}$
b . .	0·0639	0·0417	0·0089	0·2320
f . .	1·168	0·899	5·208	4·152
g . .	2·687	1·497	8·035	12·25
$\varepsilon=bg/f\beta^2$.	$3{\cdot}56\times10^{-2}$	$1{\cdot}68\times10^{-2}$	$2{\cdot}97\times10^{-4}$	$7{\cdot}33\times10^{-2}$

other choices of l and τ the corresponding values of a, b, f and g can be derived from equations (3.5). It will be observed in these equations that b is independent of the choice of l and τ as also are g/f and

$$\epsilon=\frac{bg}{\beta^2 f}=\frac{\gamma^2 T}{\rho c\beta^2}. \tag{3.12}$$

For theoretical investigations at high frequencies (Chadwick and Sneddon 1958) it is desirable to choose

$$\tau=\frac{1}{\omega^*}=\frac{\kappa}{v_P^2} \tag{3.13}$$

as the unit of time, where

$$v_P=\left(\frac{\lambda+2\mu}{\rho}\right)^{\frac{1}{2}} \tag{3.14}$$

is the velocity of pure P-waves in the solid (Bullen 1947, p. 74). The unit of length can then be taken to be

$$l=\frac{v_p}{\omega^*}. \tag{3.15}$$

With this choice of fundamental units

$$a=\beta^2, \qquad b=\frac{\gamma T}{\mu}, \qquad f=1, \qquad g=\frac{\gamma}{\rho c}, \tag{3.16}$$

where $\beta=v_p/v_s$, the ratio of the velocity of pure P-waves to that of S-waves in the solid. Thus

$$a=\beta^2=\frac{\lambda+2\mu}{\mu}=\frac{2(1-\nu)}{1-2\nu}, \tag{3.17}$$

where ν denotes Poisson's ratio of the solid. The values of b are the same as those given in Table I and the value of g is merely that of g/f obtained from Table I. The values of ϵ are unaltered. As we have said, this system of units is of value in some investigations but it should be borne in mind that ω^* is, for most materials, much higher than the highest frequency obtainable by using ultrasonic techniques and only about one-hundredth of the cut-off frequency ω_c of the Debye spectrum (Brillouin 1938, p. 324). For instance, for iron, $\omega^*=1{\cdot}75\times10^{12}$ sec.$^{-1}$ while $\omega_c=9{\cdot}95\times10^{13}$ sec.$^{-1}$; this means that the unit of length is $3{\cdot}31\times10^{-7}$ cm. so that the system is not of much interest in the discussion of engineering problems except possibly for very high frequency phenomena. For the values of these constants for aluminium, copper and lead the reader is referred to Table I of Chadwick and Sneddon (1958).

In practical problems still another system of units may be employed. For instance, if we are considering the propagation of thermal stress in a plate of thickness 1 metre, it is desirable to take $l=10^2$ cm. and we may choose

$$\tau=(10^2/v_s)\text{ sec.}, \tag{3.18}$$

where the velocity of shear waves, v_s is expressed in cm. per sec. With this choice of units we may write the equations of thermoelasticity in the forms

$$\tau_{pq,\,q}+X_p=\ddot{u}_p, \tag{3.19}$$

$$\tau_{pq}=[(\beta^2-2)\Delta-b\theta]\delta_{pq}+2\gamma_{pq}, \tag{3.20}$$

$$\chi(\nabla^2\theta+\Theta)=\dot{\theta}+\frac{g}{f}\Delta, \tag{3.21}$$

where

$$\chi = f^{-1} = \kappa v_s^{-1} 10^{-2}, \qquad \frac{g}{f} = \frac{\gamma}{\rho c}, \qquad b = \frac{\gamma T}{\mu} \tag{3.22}$$

all the physical quantities being measured in c.g.s. units. For the metals considered in Table I we get the values of χ and τ shown in Table II; in this system of units b has the same set of values as in Table I and the values of g/f can be obtained by dividing the fourth row of Table I by the

TABLE II

	Aluminium	Copper	Iron	Lead
τ (secs.) .	$3{\cdot}215 \times 10^{-4}$	$4{\cdot}654 \times 10^{-4}$	$3{\cdot}072 \times 10^{-4}$	$1{\cdot}427 \times 10^{-3}$
χ . .	$2{\cdot}750 \times 10^{-8}$	$5{\cdot}172 \times 10^{-8}$	$5{\cdot}887 \times 10^{-9}$	$3{\cdot}429 \times 10^{-8}$

third row. In calculating τ and χ in Table II we have again assumed that $T = 293°\text{K}$.

II. The Stresses Produced in an Infinite Elastic Solid by Uneven Heating

4. *The Solution of the Basic Equations*

We shall begin by considering the distribution of stress in an infinite elastic body containing heat sources, *i.e.* we shall consider the solution of equations (3.1), (3.2) and (3.3) for $-\infty < (x_1, x_2, x_3) < \infty$ and known functions Θ. It will be assumed that the solid is free from body forces so that $X_i = 0$, $(i = 1, 2, 3)$ and that the temperature and all the components of stress and displacement vanish as $x_1^2 + x_2^2 + x_3^2 \to \infty$ or as $t \to \infty$.

To solve this set of partial differential equations we introduce the four-dimensional Fourier transform of each of the physical quantities occurring in the basic equations. The Fourier transform of a function $f(x_1, x_2, x_3, t)$ is defined by the equation

$$\bar{f}(\xi_1, \xi_2, \xi_3, \omega) = \frac{1}{4\pi^2} \int_{E_4} f(x_1, x_2, x_3, t) \exp\left[i(x_p \xi_p + \omega t)\right] d\mathbf{x}, \tag{4.1}$$

where $d\mathbf{x} = dx_1 dx_2 dx_3 dt$ and E_4 denotes the entire $x_1 x_2 x_3 t$-space. If we multiply both sides of equations (3.1), (3.2) and (3.3) by $\exp\left[i(\xi_p x_p + \omega t)\right]$ and integrate over E_4 then, making use of the results

$$\frac{1}{4\pi^2} \int_{E_4} \left(\frac{\partial f}{\partial x_j}, \frac{\partial^2 f}{\partial t^2} \right) \exp\left[i(\xi_p x_p + \omega t)\right] d\mathbf{x} = -(i\xi_j, \omega^2) \bar{f} \tag{4.2}$$

(Sneddon 1951, p. 27) we find that these partial differential equations are equivalent to the set of algebraic equations

$$-i\xi_q\bar{\tau}_{pq} = -a\omega^2\bar{u}_p, \tag{4.3}$$

$$\bar{\tau}_{pq} = -[i(\beta^2-2)\xi_r\bar{u}_r + b\bar{\theta}]\delta_{pq} - i(\xi_q\bar{u}_p + \xi_p\bar{u}_q), \tag{4.4}$$

$$-\xi^2\bar{\theta} + \bar{\Theta} = -if\omega\bar{\theta} - g\omega\xi_q\bar{u}_q, \tag{4.5}$$

where $\xi^2 = \xi_q\xi_q$, from which we may obtain expressions for the Fourier transforms of the temperature and of the components of the stress tensor and the displacement vector in terms of the Fourier transform of the source function Θ.

From equation (4.5) we find that

$$\bar{\theta} = \frac{g\omega\xi_q\bar{u}_q}{\xi^2 - if\omega} + \frac{\bar{\Theta}}{\xi^2 - if\omega}. \tag{4.6}$$

If we substitute from this equation into equation (4.4) and insert the resulting expression for $\bar{\tau}_{pq}$ into equation (4.3) we obtain the equation

$$(\xi^2 - a\omega^2)\bar{u}_p + \left\{(\beta^2 - 1) - \frac{ibg\omega}{\xi^2 - if\omega}\right\}\xi_p\xi_q\bar{u}_q = \frac{ib\xi_p\bar{\Theta}}{\xi^2 - if\omega}$$

which may be solved to give

$$\bar{u}_p = \frac{ib\xi_p\bar{\Theta}}{(\xi^2 - if\omega)(\beta^2\xi^2 - a\omega^2) - ibg\omega\xi^2}. \tag{4.7}$$

Substituting this expression back in equation (4.6) we find that $\bar{\theta}$ is given by the equation

$$\bar{\theta} = \frac{(\beta^2\xi^2 - a\omega^2)\bar{\Theta}}{(\xi^2 - if\omega)(\beta^2\xi^2 - a\omega^2) - ibg\omega\xi^2},$$

which may be written in the form

$$\bar{\theta} - \bar{\theta}^c = \frac{ibg\omega\xi^2\bar{\Theta}}{(\xi^2 - if\omega)[(\xi^2 - if\omega)(\beta^2\xi^2 - a\omega^2) - ibg\omega\xi^2]}, \tag{4.8}$$

where

$$\bar{\theta}^c = \frac{\bar{\Theta}}{\xi^2 - if\omega} \tag{4.9}$$

denotes the Fourier transform of the "classical" solution. Substituting from equations (4.7) and (4.8) into (4.4) we obtain the expression

$$\bar{\tau}_{pq} = -b\bar{\theta}^c\delta_{pq} + \frac{b[(\beta^2-2)\xi^2\delta_{pq} + 2\xi_p\xi_q]\bar{\Theta} - ib^2g\omega\xi^2(\xi^2 - if\omega)^{-1}\bar{\Theta}\delta_{pq}}{(\xi^2 - if\omega)(\beta^2\xi^2 - a\omega^2) - ibg\omega\xi^2} \tag{4.10}$$

for the Fourier transforms of the components of the stress tensor. It is also easily shown from equation (4.7) that the Fourier transform of the dilatation is given by the equation

$$\bar{\Delta}=\frac{b\bar{\Theta}\xi^2}{(\xi^2-if\omega)(\beta^2\xi^2-a\omega^2)-ibg\omega\xi^2}. \tag{4.11}$$

Once the Fourier transform of a quantity has been determined the quantity itself may be calculated by means of Fourier's theorem

$$f(x_1, x_2, x_3, t)=\frac{1}{4\pi^2}\int_{W_4}\bar{f}(\xi_1, \xi_2, \xi_3, \omega)\exp\left[-i(\xi_p x_p+\omega t)\right]dW \tag{4.12}$$

(Sneddon 1951, p. 45), where $dW=d\xi_1 d\xi_2 d\xi_3 d\omega$ and W_4 is the entire $\xi_1\xi_2\xi_3\omega$-space. In this way we obtain the equation

$$u_p=\frac{b}{4\pi^2}\int_{W_4}\frac{i\xi_p\bar{\Theta}\exp\left[-i(\xi_p x_p+\omega t)\right]dW}{(\xi^2-if\omega)(\beta^2\xi^2-a\omega^2)-ibg\omega\xi^2} \tag{4.13}$$

by means of which we may calculate the components of the displacement vector, and the equation

$$\tau_{pq}=-b\theta^c\delta_{pq}+\frac{b}{4\pi^2}\int_{W_4}\frac{\left[(\beta^2-2)\xi^2\delta_{pq}+2\xi_p\xi_q-i\omega bg\xi^2(\xi^2-if\omega)^{-1}\delta_{pq}\right]\bar{\Theta}e^{-i(\xi_p x_p+\omega t)}}{(\xi^2-if\omega)(\beta^2\xi^2-a\omega^2)-ibg\omega\xi^2}dW \tag{4.14}$$

which determines the components of the stress tensor. The temperature within the solid is given by the equation

$$\theta-\theta^c=\frac{bg}{4\pi^2}\int_{W_4}\frac{i\omega\xi^2\bar{\Theta}\exp\left[-i(\xi_p x_p+\omega t)\right]dW}{\left[(\xi^2-if\omega)(\beta^2\xi^2-a\omega^2)-ibg\omega\xi^2\right](\xi^2-if\omega)}, \tag{4.15}$$

where the "classical" temperature θ^c has the form

$$\theta^c=\frac{1}{4\pi^2}\int_{W_4}\frac{\bar{\Theta}}{\xi^2-if\omega}\exp\left[-i(\xi_p x_p+\omega t)\right]dW. \tag{4.16}$$

5. *The Steady State Solution*

We shall consider the case in which the heat in the solid is generated at a constant rate so that the basic equations are each independent of the time variable t. The equations of equilibrium can be solved by the method of the previous section but it is a simple matter to derive the equilibrium solutions from the general equations of the previous section. When the source function Θ is independent of the time we may write

$$\Theta(x_1, x_2, x_3, t)=\vartheta(x_1, x_2, x_3), \tag{5.1}$$

so that

$$\bar{\Theta}(\xi_1, \xi_2, \xi_3, \omega) = (2\pi)^{\frac{1}{2}} \bar{\vartheta}(\xi_1, \xi_2, \xi_3)\delta(\omega), \tag{5.2}$$

where $\delta(\omega)$ is the Dirac delta function of argument ω and $\bar{\vartheta}$ is the three-dimensional Fourier transform of ϑ defined by the equation

$$\bar{\vartheta} = \frac{1}{(2\pi)^{\frac{3}{2}}} \int_{E_3} \vartheta \exp(i\xi_p x_p) dV \tag{5.3}$$

in which $dV = dx_1 dx_2 dx_3$ and E_3 denotes the whole $x_1x_2x_3$-space. Substituting from equation (5.2) into equation (4.13) we find that, in the case of steady heat flow, the components of the displacement vector are given by the equation

$$u_p = \frac{b}{(2\pi)^{\frac{3}{2}}} \int_{W_3} \frac{i\xi_p}{\beta^2 \xi^4} \bar{\vartheta} \exp(-i\xi_p x_p) d\xi, \tag{5.4}$$

where $d\xi = d\xi_1 d\xi_2 d\xi_3$ and W_3 is the entire $\xi_1\xi_2\xi_3$-space. Similarly from equation (4.14) we obtain the equation

$$\tau_{pq} = \frac{2b}{\beta^2 (2\pi)^{\frac{3}{2}}} \int_{W_3} \frac{\xi_p \xi_q - \xi^2 \delta_{pq}}{\xi^4} \bar{\vartheta} \exp(-i\xi_p x_p) d\xi. \tag{5.5}$$

As an example of the use of these equations, we shall consider the case in which the only source of heat in the solid is a point source at the origin which is generating heat at a constant rate. In these circumstances we may write

$$\vartheta = \vartheta_0 \delta(x_1)\delta(x_2)\delta(x_3),$$

where ϑ_0 is a constant. It follows immediately from the definition of the Dirac delta function that

$$\bar{\vartheta} = (2\pi)^{-\frac{3}{2}} \vartheta_0. \tag{5.6}$$

Substituting from equation (5.6) into equation (5.4) we find that

$$u_p = -\frac{b\vartheta_0}{8\pi^3 \beta^2} \cdot \frac{\partial I}{\partial x_p},$$

where

$$I = \int_{W_3} \frac{e^{-ix_p \xi_p}}{\xi^4} d\xi.$$

It is a simple matter to show that

$$u_p = \frac{b\vartheta_0}{8\pi^2 \beta^2} \left(\frac{x_p}{r}\right), \tag{5.7}$$

where $r^2 = x_1^2 + x_2^2 + x_3^2$. Expressions for the stress components may now

be obtained by differentiating this equation and so a full description can be given of the distribution of stress in the solid.

The solution of the general problem may be obtained from equation (5.7) by the principle of superposition. We find that

$$u_p = \frac{b}{8\pi^2\beta^2}\int_{E_3'} \frac{(x_p - x_p')\vartheta(x_1', x_2', x_3')}{R}\,dV', \tag{5.8}$$

where $dV' = dx_1'dx_2'dx_3'$ and E_3' denotes the whole $x_1'x_2'x_3'$-space. The distance R is defined by the equation

$$R^2 = (x_1 - x_1')^2 + (x_2 - x_2')^2 + (x_3 - x_3')^2. \tag{5.9}$$

The result (5.8) could also have been obtained from equation (5.4) by means of the convolution theorem for three-dimensional Fourier transforms.

6. *The Two-dimensional Problem*

The solution appropriate to a two-dimensional problem of plane strain in which the displacement vector at the point (x_1, x_2) has components (u_1, u_2) and the state of stress is uniquely determined by the three components $\tau_{\gamma\gamma'}$ $(\gamma, \gamma' = 1, 2)$, may be obtained from equation (4.13) and (4.14) by assuming that Θ is a function of x_1, x_2 and t only. We then find that the $\bar{\Theta}$ occurring in these equations should be replaced by the expression

$$(2\pi)^{\frac{1}{2}}\delta(\xi_3)\bar{\Theta}, \tag{6.1}$$

where $\bar{\Theta}(\xi_1, \xi_2, \omega)$ is now defined to be the three-dimensional Fourier transform of the source function $\Theta(x_1, x_2, t)$, so that

$$\bar{\Theta}(\xi_1, \xi_2, \omega) = \frac{1}{(2\pi)^{\frac{3}{2}}}\int_{S_3} \Theta(x_1, x_2, x_3, t)e^{i(x_1\xi_1 + x_2\xi_2 + \omega t)}\,d\mathrm{S}, \tag{6.2}$$

where $d\mathrm{S} = dx_1dx_2dt$ and S_3 is the entire x_1x_2t-space. Inserting the expression (6.1) into equation (4.13) we obtain the equation

$$u_\gamma = \frac{b}{(2\pi)^{\frac{3}{2}}}\int_{T_3} \frac{i\xi_\gamma\bar{\Theta}(\xi_1, \xi_2, \omega)\exp\left[-i(x_1\xi_1 + x_2\xi_2 + \omega t)\right]d\mathbf{T}}{(\xi_1^2 + \xi_2^2 - if\omega)[(\xi_1^2 + \xi_2^2)\beta^2 - a\omega^2] - ibg\omega(\xi_1^2 + \xi_2^2)}, \tag{6.3}$$

where $\gamma = 1, 2$, $d\mathbf{T} = d\xi_1d\xi_2d\omega$ and T_3 is the entire $\xi_1\xi_2\omega$-space.

In a similar way we can show that the components of stress are given by the tensor equation

$$\tau_{\gamma\gamma'} = \frac{b}{(2\pi)^{\frac{3}{2}}}\int_{T_3} \frac{[(\beta^2 - 2)(\xi_1^2 + \xi_2^2)\delta_{\gamma\gamma'} + 2\xi_\gamma\xi_{\gamma'} - i\omega bg(\xi_1^2 + \xi_2^2)(\xi_1^2 + \xi_2^2 - if\omega)^{-1}\delta_{\gamma\gamma'}]\bar{\Theta}e^{-i\chi}}{(\xi_1^2 + \xi_2^2 - if\omega)[\beta^2(\xi_1^2 + \xi_2^2) - a\omega^2] - i\omega bg(\xi_1^2 + \xi_2^2)}\,d\mathbf{T}$$

$$- \frac{b\delta_{\gamma\gamma'}}{(2\pi)^{\frac{3}{2}}}\int_{T_3} \frac{\bar{\Theta}e^{-i\chi}\,d\mathbf{T}}{\xi_1^2 + \xi_2^2 - if\omega}, \tag{6.4}$$

where $\gamma, \gamma' = 1, 2$, and $\chi = x_1\xi_1 + x_2\xi_2 + \omega t$.

7. *Axially Symmetrical Problems*

If we choose the axis of symmetry of the problems to be the z-axis then we need only consider the equations (3.6) to (3.11). Substituting from equations (3.6), (3.7) and (3.8) into equations (3.9) and (3.10) we find that, in the absence of body forces

$$\beta^2\left(\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}-\frac{u}{r^2}\right)+\frac{\partial^2 u}{\partial z^2}+(\beta^2-1)\frac{\partial^2 w}{\partial r\partial z}-b\frac{\partial\theta}{\partial r}=a\frac{\partial^2 u}{\partial t^2}, \tag{7.1}$$

$$\frac{\partial^2 w}{\partial r^2}+\frac{1}{r}\frac{\partial w}{\partial r}+\beta^2\frac{\partial^2 w}{\partial z^2}+(\beta^2-1)\frac{\partial}{\partial z}\left(\frac{\partial u}{\partial r}+\frac{u}{r}\right)-b\frac{\partial\theta}{\partial z}=a\frac{\partial^2 w}{\partial t^2}. \tag{7.2}$$

To solve these equations we introduce the transforms

$$\bar{u}=\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\exp\{i(\zeta z+\omega t)\}dzdt\int_0^{\infty} rJ_1(\xi r)u\,dr, \tag{7.3}$$

$$(\bar{w},\ \bar{\theta})=\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\exp\{i(\zeta z+\omega t)\}dzdt\int_0^{\infty} rJ_0(\xi r)(w,\ \theta)\,dr. \tag{7.4}$$

If we multiply both sides of equations (7.1) by $(2\pi)^{-1}\exp\{i(\zeta z+\omega t)\}\times rJ_1(\xi r)$ and both sides of equation (7.2) by $(2\pi)^{-1}\exp\{i(\zeta z+\omega t)\}rJ_0(\xi r)$ and, in both cases, integrate over the whole rzt-space then, making use of the results

$$\int_0^{\infty} r\left(\frac{\partial^2\phi}{\partial r^2}+\frac{1}{r}\frac{\partial\phi}{\partial r}-\frac{\nu^2\phi}{r}\right)J_\nu(\xi r)dr=-\xi^2\int_0^{\infty} r\phi J_\nu(\xi r)dr,$$

$$\int_0^{\infty} r\left(\frac{\partial\phi}{\partial r}+\frac{\phi}{r}\right)J_0(\xi r)dr=\xi\int_0^{\infty} r\phi J_1(\xi r)dr,$$

$$\int_0^{\infty} r\frac{\partial\phi}{\partial r}J_1(\xi r)dr=-\xi\int_0^{\infty} r\phi J_0(\xi r)dr$$

(Sneddon 1951, p. 61), we find that this pair of partial differential equations is equivalent to the pair of algebraic equations

$$(\beta^2\xi^2+\zeta^2-a\omega^2)\bar{u}-i(\beta^2-1)\xi\zeta\bar{w}=b\xi\bar{\theta},$$

$$i(\beta^2-1)\xi\zeta\bar{u}+(\xi^2+\beta^2\zeta^2-a\omega^2)\bar{w}=ib\zeta\bar{\theta}.$$

Solving these equations we find that

$$(\bar{u},\ \bar{w})=\frac{b\bar{\theta}(\xi,\ i\zeta)}{\beta^2(\xi^2+\zeta^2)-a\omega^2}. \tag{7.5}$$

If we now multiply both sides of equation (3.11) by

$$(2\pi)^{-1} \exp\,[i(\zeta z+\omega t)]rJ_0(\xi r)$$

and integrate over the whole rzt-space, we find that

$$(\xi^2+\zeta^2-if\omega)\bar{\theta}-i\omega g(\xi\bar{u}-i\zeta\bar{w})=\bar{\Theta}, \tag{7.6}$$

where

$$\bar{\Theta}=\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\exp\,[i(\zeta z+\omega t)]dzdt\int_0^{\infty}r\Theta J_0(\xi r)dr. \tag{7.7}$$

Solving the algebraic equations (7.5) and (7.6) for the unknowns $\bar{u}$, $\bar{w}$, $\bar{\theta}$ in terms of the known quantity $\bar{\Theta}$ we find the expressions

$$\frac{\bar{u}}{b\xi}=\frac{\bar{w}}{ib\zeta}=\frac{\bar{\theta}}{\beta^2(\xi^2+\zeta^2)-a\omega^2}=\frac{\bar{\Theta}}{[\beta^2(\xi^2+\zeta^2)-a\omega^2](\xi^2+\zeta^2-if\omega)-i\omega bg(\xi^2+\zeta^2)} \tag{7.8}$$

for the transforms of the components of the displacement vector and of the temperature θ. If we invert the equations (7.8) by the appropriate theorems for Fourier and Hankel transforms (Sneddon 1951, pp. 44 and 52) we find for the components of the displacement vector

$$u=\frac{b}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\exp\,[-i(\zeta z+\omega t)]d\zeta d\omega\int_0^{\infty}\frac{\xi^2\bar{\Theta}J_1(\xi r)d\xi}{D(\xi,\,\zeta,\,\omega)}, \tag{7.9}$$

$$w=\frac{ib}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\exp\,[-i(\zeta z+\omega t)]d\zeta d\omega\int_0^{\infty}\frac{\xi\zeta\bar{\Theta}J_0(\xi r)d\xi}{D(\xi,\,\zeta,\,\omega)}, \tag{7.10}$$

and for the temperature variation

$$\theta=\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\exp\,[-i(\zeta z+\omega t)]d\zeta d\omega\int_0^{\infty}\frac{\xi[\beta^2(\xi^2+\zeta^2)-a\omega^2]\bar{\Theta}J_0(\xi r)d\xi}{D(\xi,\,\zeta,\,\omega)}, \tag{7.11}$$

where

$$D(\xi,\,\zeta,\,\omega)=[\beta^2(\xi^2+\zeta^2)-a\omega^2](\xi^2+\zeta^2-if\omega)-i\omega bg(\xi^2+\zeta^2). \tag{7.12}$$

In the steady state case

$$\Theta=\vartheta(r,\,z)$$

so that

$$\bar{\Theta}=(2\pi)^{\frac{1}{2}}\bar{\vartheta}(\xi,\,\zeta)\delta(\omega). \tag{7.13}$$

Substituting from equation (7.13) into equations (7.11) and (7.12) we obtain the steady state solutions

$$u=\frac{b}{(2\pi)^{\frac{1}{2}}\beta^2}\int_{-\infty}^{\infty}e^{-i\zeta z}d\zeta\int_0^{\infty}\frac{\xi^2\bar{\vartheta}(\xi,\,\zeta)}{(\xi^2+\zeta^2)^2}J_1(\xi r)d\xi, \tag{7.14}$$

$$w=\frac{ib}{(2\pi)^{\frac{1}{2}}\beta^2}\int_{-\infty}^{\infty}\zeta e^{-i\zeta z}d\zeta\int_0^{\infty}\frac{\xi\bar{\vartheta}(\xi,\zeta)}{(\xi^2+\zeta^2)^2}J_0(\xi r)d\xi, \tag{7.15}$$

$$\theta=\frac{1}{(2\pi)^{\frac{1}{2}}}\int_{-\infty}^{\infty}e^{-i\zeta z}d\zeta\int_0^{\infty}\frac{\xi\bar{\vartheta}(\xi,\zeta)}{\xi^2+\zeta^2}J_0(\xi r)d\xi. \tag{7.16}$$

8. *The Quasi-static Solution*

If we consider problems in which the c.g.s. system of units provides natural units of length and time, *i.e.* if the centimetre is the typical distance and the second the typical time then it is obvious from Table I that the constant a is very much smaller than the other constants b, g and f occurring in the dimensionless equations (3.1) to (3.3). It follows, by expanding the integrand in equation (4.13) in ascending powers of a, that the approximate solution

$$u_p=\frac{b}{4\pi^2\beta^2}\int_{W_4}\frac{i\xi_p\bar{\Theta}\exp\left[-i(\xi_p x_p+\omega t)\right]dW}{\xi^2(\xi^2-if_1\omega)}$$
$$+\frac{ab}{4\pi^2\beta^4}\int_{W_4}\frac{i\xi_p\omega^2(\xi^2-if\omega)\bar{\Theta}\exp\left[-i(\xi_p x_p+\omega t)\right]dW}{\xi^4(\xi^2-if_1\omega)^2} \tag{8.1}$$

with

$$f_1=f(1+\epsilon) \tag{8.2}$$

will give a very accurate description of the displacement field in the elastic body. Because a is so very small we may in most cases take it to be zero and describe the displacement field by the quasi-static solution

$$u_p^{(0)}=\frac{b}{4\pi^2\beta^2}\int_{W_4}\frac{i\xi_p\bar{\Theta}\exp\left[-i(\xi_p x_p+\omega t)\right]dW}{\xi^2(\xi^2-if_1\omega)}, \qquad (p=1,2,3). \tag{8.3}$$

Similarly equations (4.14) and (4.15) may be approximated to by the equations

$$\tau_{pq}^{(0)}+b\theta^c\delta_{pq}=\frac{b}{4\pi^2\beta^2}\int_{W_4}\frac{[(\beta^2-2)\xi^2\delta_{pq}+2\xi_p\xi_q-ib\omega g\xi^2(\xi^2-if\omega)^{-1}\delta_{pq}]\bar{\Theta}e^{-i(\xi_l x_l+\omega t)}}{\xi^2(\xi^2-if_1\omega)}dW, \tag{8.4}$$

$$\theta^{(0)}-\theta^c=\frac{bg}{4\pi^2\beta^2}\int_{W_4}\frac{i\omega\xi^2\bar{\Theta}\exp\left[-i(\xi_p x_p+\omega t)\right]dW}{(\xi^2-if_1\omega)(\xi^2-if\omega)}, \tag{8.5}$$

where the "classical temperature" θ^c has the form (4.16).

The corresponding approximation to the exact solution (6.3) of the two-dimensional problem is given by

$$u_\gamma^{(0)}=\frac{b}{(2\pi)^{\frac{3}{2}}\beta^2}\int_{\pi_3}\frac{i\xi_\gamma\bar{\Theta}\exp\left[-i(x_1\xi_1+x_2\xi_2+\omega t)\right]d\mathbf{T}}{(\xi_1^2+\xi_2^2)(\xi_1^2+\xi_2^2-if_2\omega)}, \qquad (\gamma=1,2,) \tag{8.6}$$

where, now, $\bar{\Theta}$ is defined by equation (6.2). For this solution the stress components are given by the equation

$$\tau^{(0)}_{\gamma\gamma'} = -\frac{b\delta_{\gamma\gamma'}}{(2\pi)^{\frac{3}{2}}}\int_{T_3}\frac{\bar{\Theta}\exp\left[-i(x_1\xi_1+x_2\xi_2+\omega t)\right]d\mathbf{T}}{\xi_1^2+\xi_2^2-if\omega}$$

$$+\frac{b}{(2\pi)^{\frac{3}{2}}\beta^2}\int_{T_3}\frac{[(\beta^2-2)(\xi_1^2+\xi_2^2)\delta_{\gamma\gamma'}+2\xi_\gamma\xi_{\gamma'}-i\omega bg(\xi_1^2+\xi_2^2)(\xi_1^2+\xi_2^2-if\omega)^{-1}\delta_{\gamma\gamma'}]}{(\xi_1^2+\xi_2^2)(\xi_1^2+\xi_2^2-if_1\omega)}$$

$$\times\bar{\Theta}\exp\left[-i(x_1\xi_1+x_2\xi_2+\omega t)\right]d\mathbf{T}. \tag{8.7}$$

The quasi-static solution of the axially symmetrical problem is found by putting $a=0$ in equations (7.9) to (7.12). It is given by the equations

$$u^{(0)} = \frac{b}{2\pi\beta^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}e^{-i(\zeta z+\omega t)}d\zeta d\omega\int_0^{\infty}\frac{\xi^2\bar{\Theta}J_1(\xi r)d\xi}{(\xi^2+\zeta^2)(\xi^2+\zeta^2-if_1\omega)}, \tag{8.8}$$

$$w^{(0)} = \frac{ib}{2\pi\beta^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}e^{-i(\zeta z+\omega t)}d\zeta d\omega\int_0^{\infty}\frac{\xi\zeta\bar{\Theta}J_0(\xi r)d\xi}{(\xi^2+\zeta^2)(\xi^2+\zeta^2-if_1\omega)}, \tag{8.9}$$

$$\theta^{(0)} = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}e^{-i(\zeta z+\omega t)}d\zeta d\omega\int_0^{\infty}\frac{\xi\bar{\Theta}J_0(\xi r)d\xi}{\xi^2+\zeta^2-if_1\omega}. \tag{8.10}$$

The quantity $\bar{\Theta}$ occurring in these equations is defined by equation (7.7). It will be observed that in this case the temperature variation θ has the same form as it has when it is governed by the simple equation for the conduction of heat; the only difference is that the diffusion parameter f is replaced by the parameter f_1 defined by equation (8.2).

9. *Solutions of Special Problems*

In this section we shall discuss the application of these general formulæ to the solution of certain special problems.

(i) *The Stress due to a Periodic Line Source*

We shall begin by considering the stress distribution arising from a line source of periodic strength which lies along the x_3-axis. If the source is of frequency Ω_0 then we have a two-dimensional problem in which

$$\Theta = F\delta(x_1)\delta(x_2)e^{i\Omega t}, \tag{9.1}$$

where, in the units of § 3, $\Omega=\Omega_0\tau$ and F is a constant. As a result of a simple integration we see that

$$\bar{\Theta} = \frac{F}{(2\pi)^{\frac{1}{2}}}\delta(\omega+\Omega). \tag{9.2}$$

Substituting from equation (9.2) into equation (6.3) and performing the ω-integration we find that

$$u_y = -\frac{bFx_y e^{i\Omega t}}{4\pi^2 r}\frac{\partial I_1}{\partial r}, \tag{9.3}$$

where $r^2 = x_1 + x_2^2$ and

$$I_1 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{e^{-i(\xi_1 x_1 + \xi_2 x_2)} d\xi_1 d\xi_2}{(\xi_1^2 + \xi_2^2 + if\Omega)[\beta^2(\xi_1^2 + \xi_2^2) - a\Omega^2] + ibg\Omega(\xi_1^2 + \xi_2^2)}$$

$$= 2\pi \int_0^{\infty} \frac{\rho J_0(\rho r) d\rho}{(\rho^2 + if\Omega)(\beta^2\rho^2 - a\Omega^2) + ibg\Omega\rho^2}.$$

If ρ_1^2 and ρ_2^2 are the roots of the quadratic equation

$$\rho^4 + (a\Omega^2/\beta^2 - if_1\Omega)\rho^2 - iaf\Omega^3/\beta^2 = 0 \tag{9.4}$$

in ρ^2 then

$$u_y = \frac{bFx_y e^{i\Omega t}}{2\pi\beta^2 r} I_2, \tag{9.5}$$

where

$$I_2 = \int_0^{\infty} \frac{\rho^2 J_1(\rho r) d\rho}{(\rho^2 + \rho_1^2)(\rho^2 + \rho_2^2)} \tag{9.6}$$

$$= \frac{1}{\rho_2^2 - \rho_1^2}\left\{\int_0^{\infty} \frac{\rho^2 J_1(\rho r) d\rho}{\rho^2 + \rho_1^2} - \int_0^{\infty} \frac{\rho^2 J_1(\rho r) d\rho}{\rho^2 + \rho_2^2}\right\}. \tag{9.7}$$

Similarly if we put

$$\Theta = \frac{F}{2\pi r}\delta(r) e^{i\Omega t}, \tag{9.1a}$$

that is

$$\bar{\Theta} = F\delta(\omega + \Omega)\delta(\zeta) \tag{9.2a}$$

in (7.9) we find that $w = 0$ and

$$u = \frac{bFe^{i\Omega t}}{2\pi\beta^2} I_2 \tag{9.5a}$$

in agreement with equation (9.6).

Using a well-known formula in the theory of Bessel functions (Watson 1944, p. 434) we find that

$$\int_0^{\infty} \frac{\rho^2 J_1(\rho r)}{\rho^2 + k^2} d\rho = kK_1(kr), \tag{9.8}$$

where $K_\nu(z)$ denotes the modified Bessel function of the second kind of order ν and argument z and, in terms of the first Hankel function,

$$K_\nu(z) = \tfrac{1}{2}\pi i e^{\frac{1}{2}\nu\pi i} H_\nu^{(1)}(iz)$$

(Watson 1944, p. 78). Inserting this expression for the integrals occurring in equation (9.7) we obtain the exact solution

$$u = \frac{bFe^{i\Omega t}}{2\pi\beta^2(\rho_2^2-\rho_1^2)}\{\rho_1 K_1(\rho_1 r) - \rho_2 K_1(\rho_2 r)\}. \tag{9.9}$$

The temperature variation in the solid is readily found from equation (7.11). We find that

$$\theta = \frac{Fe^{i\Omega t}}{2\pi}\left\{\frac{\rho_1^2 + a\Omega^2/\beta^2}{\rho_1^2-\rho_2^2} I_3(\rho_1) - \frac{\rho_2^2 + a\Omega^2/\beta^2}{\rho_1^2-\rho_2^2} I_3(\rho_2)\right\},$$

where

$$I_3(\rho_i) = \int_0^\infty \frac{\rho J_0(\rho r)\,d\rho}{\rho^2+\rho_i^2} = K_0(\rho_i r)$$

(Watson 1944, p. 434). Hence we find that

$$\theta = \frac{Fe^{i\Omega t}}{2\pi}\left\{\frac{\rho_1^2 + a\Omega^2/\beta^2}{\rho_1^2-\rho_2^2} K_0(\rho_1 r) - \frac{\rho_2^2 + a\Omega^2/\beta^2}{\rho_1^2-\rho_2^2} K_0(\rho_2 r)\right\}. \tag{9.10}$$

The quasi-static solution obtained by inserting the value of $\bar{\Theta}$ given by equation (9.2), (or (9.2a)), into equation (8.6), (or (8.8)), may be put in the form

$$u^{(0)} = \frac{bFe^{i\Omega t} I_4}{2\pi\beta^2},$$

where

$$I_4 = \int_0^\infty \frac{J_1(\rho r)\,d\rho}{\rho^2 + if_1\Omega}.$$

Making use of equation (9.8) and a well-known result in the theory of Bessel functions (Watson 1944, p. 391) we find that

$$I_4 = \frac{1}{if_1\Omega}\int_0^\infty J_1(\rho r)\left\{\frac{\rho^2}{\rho^2+if_1\Omega} - 1\right\} d\rho$$

$$= \frac{1}{if_1\Omega}\left\{(f_1\Omega)^{\frac{1}{2}} e^{\frac{1}{2}\pi i} K_1(r_0 e^{\frac{1}{2}\pi i}) - \frac{1}{r}\right\},$$

where

$$r_0 = r f_1^{\frac{1}{2}} \Omega^{\frac{1}{2}}. \tag{9.11}$$

Hence we have

$$u^{(0)}=\frac{Fe^{i\Omega t}}{2\pi\beta^2 if_1\Omega r}\{r_0e^{\frac{1}{2}\pi i}K_1'(r_0e^{\frac{1}{2}\pi i})-1\}. \tag{9.12}$$

Furthermore, if we substitute from equation (9.2a) into equation (8.10) we find that the quasi-static value of the temperature is

$$\theta^{(0)}=\frac{Fe^{i\Omega t}}{2\pi}\int_0^\infty\frac{\xi J_0(\xi r)d\xi}{\xi^2+if_1\Omega}.$$

Evaluating by the same formula as before (Watson 1944, p. 434) we find that

$$\theta^{(0)}=\frac{Fe^{i\Omega t}}{2\pi}K_0(r_0e^{\frac{1}{2}\pi i}). \tag{9.13}$$

Using a known relation for modified Bessel functions of the second kind (Watson 1944, p. 80) we write this equation in the form

$$\theta^{(0)}=\frac{Fe^{i\Omega t}}{2\pi}\{K_0(r_0)-\tfrac{1}{2}\pi iI_0(r_0)\}. \tag{9.14}$$

From equations (9.10) and (9.13) we find that

$$\frac{\theta-\theta^{(0)}}{\theta^{(0)}}=\frac{(\rho_1^2+a\Omega^2/\beta^2)K_0'(\rho_1r)-(\rho_2^2+a\Omega^2/\beta^2)K_0'(\rho_2r)-(\rho_1^2-\rho_2^2)K_0'(r_0e^{\frac{1}{2}\pi i})}{(\rho_1^2-\rho_2^2)K_0'(r_0e^{\frac{1}{2}\pi i})}. \tag{9.15}$$

For small values of the parameter $(a\Omega/\beta^2f)$ we can show that

$$\rho_1=(f_1\Omega)^{\frac{1}{2}}e^{\frac{1}{4}\pi i}\left\{1+\frac{i\epsilon a\Omega}{2f\beta^2}\right\},\qquad \rho_2=(1-\tfrac{1}{2}\epsilon)e^{\frac{1}{4}\pi i}\frac{a^{\frac{1}{2}}\Omega}{\beta}$$

from which it follows that

$$\frac{\rho_1^2+a\Omega^2/\beta^2}{\rho_1^2-\rho_2^2}=1-\frac{i\epsilon a\Omega}{f_1\beta^2},\qquad \frac{\rho_2^2+a\Omega^2/\beta^2}{\rho_1^2-\rho_2^2}=-\frac{i\epsilon a\Omega}{f_1\beta^2},$$

and

$$\rho_1r=r_0e^{i(\frac{1}{2}\pi+\psi)},\qquad \rho_2r=r_1e^{\frac{1}{2}\pi i},\qquad \psi=\epsilon a\Omega/2f_1\beta^2,$$

where r_0 is defined by equation (9.11) and r_1 is defined by the equation

$$r_1=(1-\tfrac{1}{2}\epsilon)\frac{a^{\frac{1}{2}}\Omega}{\beta}r. \tag{9.16}$$

It should be observed that

$$\left(\frac{r_1}{r_0}\right)^2=(1-\epsilon)\frac{a\Omega}{\beta^2f_1}$$

so that, if $a\Omega/\beta^2 f_1 \ll 1$, it follows that $r_1 \ll r_0$. Using the relation

$$K_0(ze^{im\pi}) = K_0(z) - im\pi I_0(z)$$

(Watson 1944, p. 80) we find from equation (9.15) that

$$\frac{\theta - \theta^{(0)}}{\theta^{(0)}} = \frac{i\epsilon a\Omega}{\beta^2 f_1}\left\{\frac{K_0(r_1) - K_0(r_0) - \frac{1}{2}I_0(r_0) - \frac{1}{2}\pi i[I_0(r_1) - \frac{1}{2}I_0(r_0)]}{K_0(r_0) - \frac{1}{4}\pi i I_0(r_0)}\right\}. \tag{9.17}$$

If we use the asymptotic expansions of the modified Bessel functions (Watson 1944, p. 202) we find that for large values of r

$$\left|\frac{\theta - \theta^{(0)}}{\theta^{(0)}}\right| = \frac{\epsilon a\Omega}{f_1\beta^2}\phi(r_0, r_1),$$

where

$$\phi(r_0, r_1) = \left\{\left[2\left(\frac{r_0}{r_1}\right)^{\frac{1}{2}} e^{r_1 - r_0} - 1\right]^2 + \frac{4}{\pi^2}\right\}^{\frac{1}{2}}.$$

But $r_1 \ll r_0$ so that

$$\left|\frac{\theta - \theta^{(0)}}{\theta^{(0)}}\right| \simeq \frac{\epsilon a\Omega}{f_1\beta^2}\left(1 + \frac{4}{\pi^2}\right)^{\frac{1}{2}} \tag{9.18}$$

showing that for small values of $(a\Omega/f_1\beta^2)$ the quasi-static value of the temperature is a very good approximation to the exact value.

(ii) *The Effect of a Moving Line Source*

We shall consider now the effect of a line source of heat of constant strength which is moving with uniform velocity V in a direction perpendicular to its own length. If the source remains parallel to the x_3-axis and if its velocity is along the x_1-axis we may write, in the notation of equation (6.2)

$$\Theta = F\delta(x_2)\delta(x_1 - pt), \tag{9.19}$$

where

$$p = V\tau/l. \tag{9.20}$$

For this function

$$\bar{\Theta}(\xi_1, \xi_2, \omega) = \frac{F}{(2\pi)^{\frac{1}{2}}}\delta(\omega + p\xi_1)$$

so that from equation (8.6) we obtain the quasi-static solution

$$u_\gamma^{(0)} = \frac{bF}{4\pi^2\beta^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{i\xi_\gamma \exp\left[-i\{(x_1 - pt)\xi_1 + x_2\xi_2\}\right]}{(\xi_1^2 + \xi_2^2)(\xi_1^2 + \xi_2^2 + if_1 p\xi_1)}d\xi_1 d\xi_2.$$

For this solution the components of stress $\tau_{11}^{(0)}$, $\tau_{22}^{(0)}$, $\tau_{12}^{(0)}$ are given by the equation

$$\tau_{\gamma\gamma'}^{(0)} = -\frac{F\delta_{\gamma\gamma'}}{4\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{\exp\left[-i\{(x_1-pt)\xi_1+x_2\xi_2\}\right]}{\xi_1^2+\xi_2^2+if p\xi_1}d\xi_1 d\xi_2$$

$$+\frac{bF}{4\pi^2\beta^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{[(\beta^2-2)(\xi_1^2+\xi_2^2)\delta_{\gamma\gamma'}+2\xi_\gamma\xi_{\gamma'}+ibgp\xi_1(\xi_1^2+\xi_2^2)(\xi_1^2+\xi_2^2+if_1p\xi_1)^{-1}\delta_{\gamma\gamma'})]}{(\xi_1^2+\xi_2^2)(\xi_1^2+\xi_2^2+if_1p\xi_1)}$$

$$\times\exp\left[-i\{(x_1-pt)\xi_1+x_2\xi_2\}\right]d\xi_1 d\xi_2. \tag{9.21}$$

For example, we find that

$$\tau_{12}^{(0)} = \frac{bF}{2\pi^2\beta^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{\xi_1\xi_2\exp\left[-i\{(x_1-pt)\xi_1+x_2\xi_2\}\right]}{(\xi_1^2+\xi_2^2)(\xi_1^2+\xi_2^2+if_1p\xi_1)}d\xi_1 d\xi_2. \tag{9.22}$$

Performing the integration with respect to ξ_1 we find that

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{\xi_1\xi_2\exp\left[-i\{(x_1-pt)\xi_1+x_2\xi_2\}\right]d\xi_1 d\xi_2}{(\xi_1^2+\xi_2^2)(\xi_1^2+\xi_2^2+if_1p\xi_1)}$$

$$=\frac{2\pi}{f_1p}\int_0^{\infty}\left\{\frac{2\xi_2}{(4\xi_2^2+p^2)^{\frac{1}{2}}}e^{-\frac{1}{2}(x_1-pt)[(4\xi_2^2+f_1^2p^2)^{\frac{1}{2}}+f_1p]}-e^{-\xi_2(x_1-pt)}\right\}\sin(\xi_2x_2)d\xi_2,$$

and it can be shown that this integral has the value

$$\frac{2\pi x_2}{f_1pR}\left\{\frac{2}{f_1p}e^{-\frac{1}{2}f_1p(x_1-pt)}K_1(\tfrac{1}{2}f_1pR)-1\right\}, \tag{9.23}$$

where R is defined by the equation

$$R^2=(x_1-pt)^2+x_2^2. \tag{9.24}$$

Substituting from equation (9.23) into equation (9.22) we find that

$$\tau_{12}^{(0)} = \frac{bx_2F}{\pi\beta^2f_1pR}\left\{\frac{2}{f_1p}e^{-\frac{1}{2}f_1p(x_1-pt)}K_1(\tfrac{1}{2}f_1pR)-1\right\}. \tag{9.25}$$

In a similar way we can establish the result

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{e^{-i(x_1-pt)\xi_1-ix_2\xi_2}}{\xi_1^2+\xi_2^2+ipf\xi_1}d\xi_1 d\xi_2 = 4\pi\int_0^{\infty}\frac{e^{-\frac{1}{2}(x_1-pt)[(4\xi_2^2+f^2p^2)^{\frac{1}{2}}+fp]}}{(4\xi_2^2+f^2p^2)^{\frac{1}{2}}}d\xi_1 d\xi_2$$

$$=\frac{2\pi(x_1-pt)}{R}K_1(\tfrac{1}{2}fpR)e^{-\frac{1}{2}fp(x_1-pt)} \tag{9.26}$$

from which we obtain the equations

$$\tau_{11}^{(0)}+\tau_{22}^{(0)} = \frac{bF}{\pi}\left\{\frac{\beta^2-1}{\beta^2}e^{-\frac{1}{2}f_1p(x_1-pt)}K_0(\tfrac{1}{2}f_1pR)-e^{-\frac{1}{2}fp(x_1-pt)}K_0(\tfrac{1}{2}fpR)\right\}$$

$$+\frac{b^2gF}{2\pi\beta^2}\left(\frac{x_1-pt}{R}\right)\{K_1(\tfrac{1}{2}fpR)e^{-\frac{1}{2}fp(x_1-pt)}-K_1(\tfrac{1}{2}f_1pR)e^{-\frac{1}{2}f_1p(x_1-pt)}\}, \tag{9.27}$$

$$\tau_{11}^{(0)}-\tau_{22}^{(0)}=\frac{2bF}{\pi\beta^2 f_1 pR}\left\{\frac{2}{f_1 p}e^{-\frac{1}{2}f_1 p(x_1-pt)}K_1(\tfrac{1}{2}f_1 pR)-\frac{1}{R}\right\} \tag{9.28}$$

by means of which we may calculate the stress components $\tau_{11}^{(0)}$ and $\tau_{22}^{(0)}$.

(iii) *The Stress due to an Impulsive Line Source*

We shall now consider the effect of an impulsive line source of strength F applied along the line $x_1=x_2=0$. This may be represented by

$$\Theta(x_1, x_2, t)=F\delta(x_1)\delta(x_2)\delta(t)$$

from which, in the notation of equation (6.2),

$$\bar{\Theta}=\frac{F}{(2\pi)^{\frac{3}{2}}}. \tag{9.29}$$

Substituting from equation (9.29) into equation (6.3) we obtain the solution

$$u_\gamma=\frac{bF}{8\pi^3}\int_{T_3}\frac{i\xi_\gamma\exp\left[-i(\xi_1x_1+\xi_2x_2+\omega t)\right]d\mathbf{T}}{\beta^2(\xi_1^2+\xi_2^2)^2-(i\omega f_1\beta^2+a\omega^2)(\xi_1^2+\xi_2^2)+iaf\omega^3}.$$

Alternatively if we use cylindrical co-ordinates then in the notation of § 7,

$$\Theta=\frac{F}{2\pi r}\delta(r)\delta(t)$$

so that by equation (7.7.)

$$\bar{\Theta}=\frac{F\delta(\zeta)}{2\pi}.$$

Inserting this expression in equations (7.9), (7.10) and (7.11) we find that $w=0$ and that u and θ are given by the equations

$$u=\frac{bF}{4\pi^2}\int_{-\infty}^{\infty}e^{-i\omega t}d\omega\int_0^\infty\frac{\xi^2J_1(\xi r)d\xi}{\mathfrak{D}(\xi,\omega)}, \tag{9.30}$$

$$\theta=\frac{F}{4\pi^2}\int_{-\infty}^{\infty}e^{-i\omega t}d\omega\int_0^\infty\frac{(\beta^2\xi^2-a\omega^2)\xi J_0(\xi r)d\xi}{\mathfrak{D}(\xi,\omega)}, \tag{9.31}$$

where

$$\mathfrak{D}=(\beta^2\xi^2-a\omega^2)(\xi^2-if\omega)-i\omega bg\xi^2. \tag{9.32}$$

If we assume that $a=0$ we find that equation (9.30) reduces to

$$u^{(0)}=\frac{bF}{2\pi\beta^2f_1}\int_0^\infty e^{-\xi^2t/f_1}J_1(\xi r)d\xi.$$

Now

$$\int_0^\infty J_1(\rho r)\mathrm{e}^{-\lambda\rho^2}d\rho = \begin{cases} 0 & \lambda < 0, \\ \frac{1}{r}(1-\mathrm{e}^{-r^2/4\lambda}), & \lambda > 0, \end{cases}$$

so that

$$u^{(0)} = \begin{cases} 0 & t < 0, \\ \frac{bF}{2\pi\beta^2 f_1 r}(1-\mathrm{e}^{-f_1 r^2/4t}), & t > 0. \end{cases}$$

Similarly, from equation (9.31) we find that approximately

$$\theta^{(0)} = \begin{cases} 0 & t < 0, \\ \frac{F}{4\pi t}\mathrm{e}^{-f_1 r^2/4t}, & t > 0. \end{cases}$$

Since we have assumed that $a=0$ we should expect these solutions to be valid if $r \ll v_s t$ (in conventional units). For very short times, *i.e.* immediately after the application of the thermal impulse these expressions would not be valid.

The corresponding components of stress for $t \gg r/v_s$ are

$$\sigma_r^{(0)} = -\frac{bF}{\pi\beta^2 f_1 r^2}(1-\mathrm{e}^{-f_1 r^2/4t}),$$

$$\sigma_\phi^{(0)} = -\frac{bF}{2\pi\beta^2 f_1 r^2}\left\{\frac{f_1 r^2}{t}\mathrm{e}^{-f_1 r^2/4t} - 2(1-\mathrm{e}^{-f_1 r^2/4t})\right\},$$

$$\sigma_z^{(0)} = -\frac{bF}{2\pi\beta^2 t}\mathrm{e}^{-f_1 r^2/4t},$$

and the shearing stress τ_{rz} is identically zero.

(iv) *Impulsive Point Source of Heat*

To illustrate further the use of the axially symmetrical solution we shall consider the effect of an impulsive point source of strength q situated at the origin. For such a source we have

$$\Theta = \frac{q}{2\pi r}\delta(r)\delta(z)\delta(t)$$

so that, in the notation of equation (7.7),

$$\bar{\Theta} = \frac{q}{2\pi}. \tag{9.33}$$

Substituting from equation (9.33) into equation (8.8) we obtain the approximate solution, valid for all but very large values of r/t,

$$u^{(0)}=\frac{bq}{4\pi\beta^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}e^{-i(\zeta z+\omega t)}d\zeta d\omega\int_0^{\infty}\frac{\xi^2J_1(\xi r)d\xi}{(\xi^2+\zeta^2)(\xi^2+\zeta^2-if_1\omega)}.$$

Performing the integration with respect to ω we find that

$$u^{(0)}=\frac{bq}{\pi\beta^2f_1}\int_0^{\infty}\xi^2J_1(\xi r)e^{-\xi^2t/f_1}d\xi\int_0^{\infty}\frac{\cos(\zeta z)e^{-\zeta^2t/f_1}d\zeta}{\xi^2+\zeta^2},$$

and then performing the integration with respect to ζ (Erdelyi *et al.* 1954, p. 15), we obtain for the radial component of the displacement the expression

$$u^{(0)}=\frac{bq}{4\beta^2f_1}\int_0^{\infty}\xi J_1(\xi r)\left\{e^{-\xi z}Erfc\left(\frac{t^{\frac{1}{2}}\xi}{f_1^{\frac{1}{2}}}-\frac{zf_1^{\frac{1}{2}}}{2t^{\frac{1}{2}}}\right)+e^{\xi z}Erfc\left(\frac{t^{\frac{1}{2}}\xi}{f_1^{\frac{1}{2}}}+\frac{zf_1^{\frac{1}{2}}}{2t^{\frac{1}{2}}}\right)\right\}d\xi$$

from which numerical values may be obtained by quadratures.

Similarly it can be shown that

$$w^{(0)}=\frac{bq}{4\beta^2f_1}\int_0^{\infty}\xi J_0(\xi r)\left\{e^{-\xi z}Erfc\left(\frac{t^{\frac{1}{2}}\xi}{f_1^{\frac{1}{2}}}-\frac{zf_1^{\frac{1}{2}}}{2t^{\frac{1}{2}}}\right)-e^{\xi z}Erfc\left(\frac{t^{\frac{1}{2}}\xi}{f_1^{\frac{1}{2}}}+\frac{zf_1^{\frac{1}{2}}}{2t^{\frac{1}{2}}}\right)\right\}d\xi.$$

III. The Stresses Produced in a Semi-infinite Solid by Uneven Heating of the Surface

10. *The Solution of the Basic Equations*

We shall now consider the case in which the solid is bounded by a plane which is free from applied stress but whose surface temperature is made to vary in a prescribed way and which has heat sources of known strength in the interior.

In the case of a semi-infinite solid the symmetry of the equations, when written in cartesian co-ordinates, is not preserved, so that it is no longer advantageous to make use of the notation $x_i(i=1, 2, 3)$ employed in Part II. In this section we shall denote the co-ordinates of a typical point of the solid by (x, y, z) and assume that the solid is bounded by the plane $z=0$, occupying the space $z>0$. If we denote the components of the displacement vector by (u, v, w) and those of the stress tensor by $\sigma_x, \sigma_y, \sigma_z, \tau_{yz}, \tau_{zx}, \tau_{xy}$ we may write equations (3.1) in the forms

$$\frac{\partial\sigma_x}{\partial x}+\frac{\partial\tau_{xy}}{\partial y}+\frac{\partial\tau_{xz}}{\partial z}=a\frac{\partial^2u}{\partial t^2}, \tag{10.1}$$

$$\frac{\partial \tau_{xy}}{\partial x}+\frac{\partial \sigma_y}{\partial y}+\frac{\partial \tau_{yz}}{\partial z}=a\frac{\partial^2 v}{\partial t^2}, \tag{10.2}$$

$$\frac{\partial \tau_{xz}}{\partial x}+\frac{\partial \tau_{yz}}{\partial y}+\frac{\partial \sigma_z}{\partial z}=a\frac{\partial^2 w}{\partial t^2} \tag{10.3}$$

(in the absence of body forces). The stress-strain relations (3.2) take the forms

$$\sigma_x=\beta^2\frac{\partial u}{\partial x}+(\beta^2-2)\frac{\partial v}{\partial y}+(\beta^2-2)\frac{\partial w}{\partial z}-b\theta, \tag{10.4}$$

$$\sigma_y=(\beta^2-2)\frac{\partial u}{\partial x}+\beta^2\frac{\partial v}{\partial y}+(\beta^2-2)\frac{\partial w}{\partial z}-b\theta, \tag{10.5}$$

$$\sigma_z=(\beta^2-2)\frac{\partial u}{\partial x}+(\beta^2-2)\frac{\partial v}{\partial y}+\beta^2\frac{\partial w}{\partial z}-b\theta, \tag{10.6}$$

$$\tau_{yz}=\frac{\partial v}{\partial z}+\frac{\partial w}{\partial y}, \tag{10.7}$$

$$\tau_{zx}=\frac{\partial w}{\partial x}+\frac{\partial u}{\partial z}, \tag{10.8}$$

$$\tau_{xy}=\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}, \tag{10.9}$$

where the temperature variation θ satisfies the equation

$$\frac{\partial^2\theta}{\partial x^2}+\frac{\partial^2\theta}{\partial y^2}+\frac{\partial^2\theta}{\partial z^2}+\text{Ⓗ}=f\frac{\partial\theta}{\partial t}+g\frac{\partial}{\partial t}\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}\right). \tag{10.10}$$

If we introduce the sets of transforms

$(\bar\sigma_x, \bar\sigma_y, \bar\sigma_z, \bar\tau_{xy}, \bar u, \bar v, \bar\theta)$

$$=\frac{1}{2\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}e^{i(\xi x+\eta y+\omega t)}dx\,dy\,dt\int_0^{\infty}(\sigma_x, \sigma_y, \sigma_z, \tau_{xy}, u, v, \theta)\sin\zeta z\,dz, \tag{10.11}$$

$$(\bar\tau_{xz}, \bar\tau_{yz}, \bar w)=\frac{1}{2\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}e^{i(\xi x+\eta y+\omega t)}dx\,dy\,dt\int_0^{\infty}(\tau_{xz}, \tau_{yz}, w)\cos\zeta z\,dz \tag{10.12}$$

then multiplying equations (10.1) and (10.2) by $\exp[i(\xi x+\eta y+\omega t)]\sin\zeta z$ and equation (10.3) by $\exp[i(\xi x+\eta y+\omega t)]\cos(\zeta z)$ and integrating over all t throughout the half space and assuming that

$$\sigma_z=0 \quad \text{when} \quad z=0 \tag{10.13}$$

we find that

$$\left.\begin{aligned} i\xi\bar\sigma_x+i\eta\bar\tau_{xy}+\zeta\bar\tau_{xz}&=a\omega^2\bar u,\\ i\xi\bar\tau_{xy}+i\eta\bar\sigma_y+\zeta\bar\tau_{yz}&=a\omega^2\bar v,\\ i\xi\bar\tau_{xz}+i\eta\bar\tau_{yz}+\zeta\bar\sigma_z&=a\omega^2\bar w.\end{aligned}\right\} \tag{10.14}$$

Similarly if we assume that

$$u = v = 0 \quad \text{when} \quad z = 0 \tag{10.15}$$

we find that the partial differential equations (10.4) to (10.9) are equivalent to the following set of six algebraic equations

$$\left.\begin{aligned}
\bar{\sigma}_x &= -i\beta^2\xi\bar{u} - i(\beta^2-2)\eta\bar{v} - (\beta^2-2)\zeta\bar{w} - b\bar{\theta},\\
\bar{\sigma}_y &= -i(\beta^2-2)\xi\bar{u} - i\beta^2\eta\bar{v} - (\beta^2-2)\zeta\bar{w} - b\bar{\theta},\\
\bar{\sigma}_z &= -i(\beta^2-2)\xi\bar{u} - i(\beta^2-2)\eta\bar{v} - \beta^2\zeta\bar{w} - b\bar{\theta},\\
\bar{\tau}_{yz} &= (\zeta\bar{v} - i\eta\bar{w}),\\
\bar{\tau}_{zx} &= (-i\xi\bar{w} + \zeta\bar{u}),\\
\bar{\tau}_{xy} &= -i(\eta\bar{u} + \xi\bar{v}).
\end{aligned}\right\} \tag{10.16}$$

With the same assumptions about the surface values of u and v we find on multiplying both sides of equation (10.10) by $\exp[i(\xi x + \eta y + \omega t)] \sin \zeta z$ and integrating that

$$\bar{\Theta} - (\xi^2 + \eta^2 + \zeta^2)\bar{\theta} + \zeta\bar{\theta}_0 = -if\omega\bar{\theta} - g\xi\omega\bar{u} - g\eta\omega\bar{v} + ig\omega\zeta\bar{w}, \tag{10.17}$$

where

$$\left.\begin{aligned}
\bar{\Theta} &= \frac{1}{2\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{i(\xi x+\eta y+\omega t)}dx\,dy\,dt\int_0^{\infty}\Theta \sin \zeta z\,dz,\\
\bar{\theta}_0 &= \frac{1}{2\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{i(\xi x+\eta y+\omega t)}dx\,dy\,dt
\end{aligned}\right\} \tag{10.18}$$

in which θ_0 denotes the surface temperature.

Solving the set of algebraic equations (10.14), (10.16) and (10.17) we find that the transforms of the components of the displacement vector are given by the equations

$$(\bar{u}, \bar{v}, \bar{w}) = (i\xi, i\eta, -\zeta)b\bar{G}, \tag{10.19}$$

where

$$\bar{G} = \frac{\bar{\Theta} + \zeta\bar{\theta}_0}{(\gamma^2 - if\omega)(\beta^2\gamma^2 - a\omega^2) - ibg\omega\gamma^2} \tag{10.20}$$

with $\gamma^2 = \xi^2 + \eta^2 + \zeta^2$. Inverting equations (10.19) by means of Fourier's theorem for multiple transforms (Sneddon 1951, p. 45) we find that the components of the displacement vector are given by the equations

$$u = \frac{b}{2\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} i\xi e^{-i(\xi x+\eta y+\omega t)}d\xi\,d\eta\,d\omega\int_0^{\infty}\bar{G}\sin(\zeta z)\,d\zeta, \tag{10.21}$$

$$v=\frac{b}{2\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} i\eta e^{-i(\xi x+\eta y+\omega t)}d\xi d\eta d\omega \int_0^{\infty}\bar{G}\sin(\zeta z)d\zeta, \tag{10.22}$$

$$w=\frac{b}{2\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-i(\xi x+\eta y+\omega t)}d\xi d\eta d\omega \int_0^{\infty}\zeta\bar{G}\cos(\zeta z)d\zeta, \tag{10.23}$$

which can be written in the form

$$(u, v, w)=-\operatorname{grad}\psi, \tag{10.24}$$

with

$$\psi=\frac{b}{2\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-i(\xi x+\eta y+\omega t)}d\xi d\eta d\omega \int_0^{\infty}\bar{G}\sin\zeta z d\zeta. \tag{10.25}$$

The equations (10.24) and (10.25) constitute the solution of the problem in which there is a source function Θ in the solid and a surface distribution of temperature θ_0 and the mechanical boundary conditions are defined by equation (10.13) and (10.15). For this solution we find that when $z=0$ the shearing stresses τ_{xz}, τ_{yz} are given by the equations

$$(\tau_{xz})_{z=0}=q_1(x, y, t)=\frac{b}{\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} i\xi e^{-i(\xi x+\eta y+\omega t)}d\xi d\eta d\omega\int_0^{\infty}\zeta\bar{G}d\zeta, \tag{10.26}$$

$$(\tau_{yz})_{z=0}=q_2(x, y, t)=\frac{b}{\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} i\eta e^{-i(\xi x+\eta y+\omega t)}d\xi d\eta d\omega\int_0^{\infty}\zeta\bar{G}d\zeta. \tag{10.27}$$

If, therefore, we wish to find a solution to the problem in which

$$\sigma_z=\tau_{xz}=\tau_{yz}=0 \quad \text{on} \quad z=0$$

we have to add to the solution (10.24) the solution of the (purely elastic) boundary value problem

$$\sigma_z=0, \qquad \tau_{xz}=-q_1, \qquad \tau_{yz}=-q_2 \quad \text{on} \quad z=0, \tag{10.28}$$

where q_1, q_2 are defined by equations (10.26) and (10.27) respectively. The solution of the boundary value problem corresponding to the conditions (10.28) has been derived previously (Eason, 1954). It should be noted that for the functions involved in this particular form of the problem

$$\frac{1}{(2\pi)^{\frac{3}{2}}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(q_1, q_2)e^{i(\xi x+\eta y+\omega t)}dxdydt=(\xi, \eta)I, \tag{10.29}$$

where

$$I(\xi, \eta, \omega)=ib\left(\frac{2}{\pi}\right)^{\frac{1}{2}}\int_0^{\infty}\frac{\zeta(\bar{\Theta}+\zeta\bar{\theta}_0)d\zeta}{(\gamma^2-if\omega)(\beta^2\gamma^2-a\omega^2)-ibg\omega\gamma^2}. \tag{10.30}$$

11. *The Steady State Solution*

We shall now consider the case in which the surface temperature does not vary with time and the rate at which heat is generated in the solid is constant so that each of the basic equations is independent of the time variable t. If

$$\Theta = q(x, y, z), \qquad \theta_0 = \vartheta_0(x, y) \tag{11.1}$$

then, in the notation of equations (10.18),

$$\bar{\Theta} = (2\pi)^{\frac{1}{2}} \bar{q} \delta(\omega)$$

where

$$\bar{q} = \frac{1}{(2\pi^3)^{\frac{1}{2}}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{i(\xi x + \eta y)} dx dy \int_0^{\infty} q \sin \zeta z dz \tag{11.2}$$

and

$$\bar{\theta} = 2\bar{\vartheta}_0 \delta(\omega)$$

where

$$\bar{\vartheta}_0 = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \vartheta_0 e^{i(\xi x + \eta y)} dx dy. \tag{11.3}$$

Therefore if we write

$$\bar{H} = \frac{2\zeta \bar{\vartheta}_0 + (2\pi)^{\frac{1}{2}} \bar{q}}{\gamma^4} \tag{11.4}$$

we find that

$$\bar{G} = \bar{H} \delta(\omega). \tag{11.5}$$

Substituting from equation (11.5) into equation (10.24) and (10.25) we obtain the steady state solution

$$(u, v, w) = -\operatorname{grad} \chi \tag{11.6}$$

where

$$\chi = \frac{b}{2\pi^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-i(\xi x + \eta y)} d\xi d\eta \int_0^{\infty} \bar{H} \sin(\zeta z) d\zeta \tag{11.7}$$

with $\bar{H}$ defined by equations (11.2), (11.3) and (11.4). This solution corresponds to the boundary conditions

$$\sigma_z = u = v = 0 \quad \text{on} \quad z = 0.$$

To obtain a solution to the problem in which

$$\sigma_z = \tau_{xz} = \tau_{yz} = 0 \quad \text{on} \quad z = 0$$

we have to add to (11.6) the solution of the statical boundary value problem of pure elasticity in which

$$\sigma_z = 0, \qquad \tau_{xz} = -s_1, \qquad \tau_{yz} = -s_2 \quad \text{on} \quad z = 0, \tag{11.8}$$

where s_1, s_2 are defined by the pair of equations

$$(s_1, s_2) = \frac{bi}{\pi^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (\xi, \eta) e^{-i(\xi x + \eta y)} d\xi d\eta \int_0^{\infty} \zeta \bar{F} d\zeta. \tag{11.9}$$

The solution of this boundary value problem is well-known (Sneddon 1951, p. 444), so that the complete solution of the steady state thermoelastic problem can be derived.

12. *The Solution of the Two-dimensional Problem*

We may obtain the solution of the two-dimensional thermoelastic equations for a half-space by substituting in equations (10.21) and (10.23) the expressions

$$\bar{\Theta} = (2\pi)^{\frac{1}{2}} \bar{q}(\xi, \zeta, \omega) \delta(\eta) \qquad \bar{\theta}_0 = \bar{\vartheta}_0(\xi, \omega) \delta(\eta),$$

where

$$\bar{q}(\xi, \zeta, \omega) = \frac{1}{(2\pi^3)^{\frac{1}{2}}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{i(\xi x + \omega t)} dx dt \int_0^{\infty} \Theta(x, z, t) \sin(\zeta z) dz, \tag{12.1}$$

and

$$\bar{\theta}_0(\xi, \omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \theta(x, 0, t) e^{i(\xi x + \omega t)} dx dt. \tag{12.2}$$

In this way we obtain the solution

$$u = \frac{b}{2\pi^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} i\xi e^{-i(\xi x + \omega t)} d\xi d\omega \int_0^{\infty} \bar{F} \sin(\zeta z) d\zeta, \tag{12.3}$$

$$w = -\frac{b}{2\pi^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-i(\xi x + \omega t)} d\xi d\omega \int_0^{\infty} \zeta \bar{F} \cos(\zeta z) d\zeta, \tag{12.4}$$

where

$$\bar{F} = \frac{(2\pi)^{\frac{1}{2}} \bar{q} + \zeta \bar{\vartheta}_0}{(\xi^2 + \zeta^2 - if\omega)(\beta^2 \xi^2 + \beta^2 \zeta^2 - a\omega^2) - ibg\omega(\xi^2 + \zeta^2)}. \tag{12.5}$$

It is immediately obvious that this solution can be written in the form

$$(u, w) = -\left(\frac{\partial \psi}{\partial x}, \frac{\partial \psi}{\partial z}\right) \tag{12.6}$$

with

$$\psi=\frac{b}{2\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}e^{-i(\xi x+\omega t)}d\xi d\omega\int_0^{\infty}\zeta\bar{F}d\zeta. \tag{12.7}$$

For this solution we find that when $z=0$, $\sigma_z=0$ and

$$\tau_{xz}=\frac{b}{2\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}i\xi e^{-i(\xi x+\omega t)}d\xi d\omega\int_0^{\infty}\zeta\bar{F}d\zeta$$

so that to obtain the solution corresponding to a boundary free from applied stress we must add to the solution (12.6) that solution of the dynamical equations which gives for $z=0$

$$\bar{\sigma}_z=0, \qquad \bar{\tau}_{xz}=-2\bar{\tau}, \tag{12.8}$$

where

$$\bar{\tau}=\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\tfrac{1}{2}(\tau_{xz})_{z=0}e^{i(\xi x+\omega t)}dxdt=\frac{bi\xi}{\pi}\int_0^{\infty}\zeta\bar{F}d\zeta. \tag{12.9}$$

The solution of this latter problem is (*cf.* Eason, 1954)

$$u=\frac{1}{2\pi(\beta^2-1)}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{\bar{\tau}e^{-|\xi|z}}{|\xi|}\{\beta^2-(\beta^2-1)|\xi|z\}e^{-i(\xi x+\omega t)}d\xi d\omega, \tag{12.10}$$

$$w=\frac{1}{2\pi(\beta^2-1)}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}i\bar{\tau}e^{-|\xi|z}\{1+(\beta^2-1)|\xi|z\}e^{-i(\xi x+\omega t)}d\xi d\omega. \tag{12.11}$$

Hence the complete solution to our two-dimensional problem is contained in the equations

$$u=\frac{b}{2\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}i\xi e^{-i(\xi x+\omega t)}d\xi d\omega\int_0^{\infty}\bar{F}\sin(\zeta z)d\zeta$$

$$+\frac{b}{2\pi^2(\beta^2-1)}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{i\xi}{|\xi|}\{\beta^2-(\beta^2-1)|\xi|z\}e^{-|\xi|z-i(\xi x+\omega t)}d\xi d\omega\int_0^{\infty}\zeta\bar{F}d\zeta, \tag{12.12}$$

$$w=\frac{b}{2\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}e^{-i(\xi x+\omega t)}d\xi d\omega\int_0^{\infty}\zeta\bar{F}\cos(\zeta z)d\zeta$$

$$-\frac{b}{2\pi^2(\beta^2-1)}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\{1+(\beta^2-1)|\xi|z\}e^{-|\xi|z-i(\xi x+\omega t)}d\xi d\omega\int_0^{\infty}\zeta\bar{F}d\zeta. \tag{12.13}$$

On the boundary $z=0$ we find that the normal displacement is

$$W=-\frac{b\beta^2}{2\pi^2(\beta^2-1)}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}e^{-i(\xi x+\omega t)}d\xi d\omega\int_0^{\infty}\zeta\bar{F}d\zeta. \tag{12.14}$$

13. *The Effect of a Periodic Line Temperature Applied to the Surface*

To illustrate the use of the above formulæ we shall consider the effect of a line distribution of temperature of magnitude P which is periodic in time applied along the line $y=z=0$ which may be represented by the formula

$$\theta(x, 0, t)=P\delta(x)e^{i\Omega t}, \tag{13.1}$$

so that from equation (12.2)

$$\bar{\vartheta}_0(\xi, \omega)=P\delta(\omega+\Omega),$$

and from equation (12.5)

$$\bar{F}(\xi, \zeta, \omega)=\frac{P\zeta\delta(\omega+\Omega)}{\beta^2(\xi^2+\zeta^2+\rho_1^2)(\xi^2+\zeta^2+\rho_2^2)}, \tag{13.2}$$

where ρ_1^2 and ρ_2^2 are the roots of the quadratic equation (9.4). For this function

$$\int_0^\infty \zeta\bar{F}(\xi, \zeta, \omega)d\zeta=\frac{\pi P\delta(\omega+\Omega)}{2(\rho_1^2-\rho_2^2)\beta^2}\{(\xi^2+\rho_1^2)^{\frac{1}{2}}-(\xi^2+\rho_2^2)^{\frac{1}{2}}\},$$

$$\int_0^\infty \zeta\bar{F}(\xi, \zeta, \omega)\cos(\zeta z)d\zeta=\frac{\pi P\delta(\omega+\Omega)}{2(\rho_1^2-\rho_2^2)\beta^2}\{(\xi^2+\rho_1^2)^{\frac{1}{2}}e^{-(\xi^2+\rho_1^2)^{\frac{1}{2}}z}-(\xi^2+\rho_2^2)^{\frac{1}{2}}e^{-(\xi^2+\rho_2^2)^{\frac{1}{2}}z}\},$$

$$\int_0^\infty \bar{F}(\xi, \zeta, \omega)\sin(\zeta z)d\zeta=\frac{\pi P\delta(\omega+\Omega)}{2(\rho_1^2-\rho_2^2)\beta^2}\{e^{-(\xi^2+\rho_1^2)^{\frac{1}{2}}z}-e^{-(\xi^2+\rho_2^2)^{\frac{1}{2}}z}\}.$$

Substituting from these expressions into equations (12.12) and (12.13) we obtain the expressions

$$u=\frac{bPe^{i\Omega t}}{2\pi(\rho_1^2-\rho_2^2)\beta^2}\int_0^\infty \xi[e^{-(\xi^2+\rho_1^2)^{\frac{1}{2}}z}-e^{-(\xi^2+\rho_2^2)^{\frac{1}{2}}z}]\sin(\xi x)d\xi$$

$$+\frac{bPe^{i\Omega t}}{2\pi(\rho_1^2-\rho_2^2)\beta^2(\beta^2-1)}\int_0^\infty [\beta^2-(\beta^2-1)\xi z][(\xi^2+\rho_1^2)^{\frac{1}{2}}-(\xi^2+\rho_2^2)^{\frac{1}{2}}]e^{-\xi z}\sin(\xi x)d\xi, \tag{13.3}$$

$$w=-\frac{bPe^{i\Omega t}}{2\pi(\rho_1^2-\rho_2^2)\beta^2}\int_0^\infty [(\xi^2+\rho_1^2)^{\frac{1}{2}}e^{-(\xi^2+\rho_1^2)^{\frac{1}{2}}z}-(\xi^2+\rho_2^2)^{\frac{1}{2}}e^{-(\xi^2+\rho_2^2)^{\frac{1}{2}}z}]\cos(x\xi)d\xi$$

$$-\frac{bPe^{i\Omega t}}{2\pi(\rho_1^2-\rho_2^2)\beta^2(\beta^2-1)}\int_0^\infty [1+(\beta^2-1)\xi z][(\xi^2+\rho_1^2)^{\frac{1}{2}}-(\xi^2+\rho_2^2)^{\frac{1}{2}}]e^{-\xi z}\cos(x\xi)d\xi. \tag{13.4}$$

Now it is readily shown from known results (Erdélyi 1954, pp. 16 and 75) that

$$\int_0^\infty \xi[e^{-(\xi^2+\rho_1^2)^{\frac{1}{2}}z} - e^{-(\xi^2+\rho_2^2)^{\frac{1}{2}}z}] \sin(\xi x)d\xi = \frac{xz}{r^2}[\rho_1^2 K_2(\rho_1 r) - \rho_2^2 K_2(\rho_2 r)],$$

$$\int_0^\infty [(\xi^2+\rho_1^2)^{\frac{1}{2}} e^{-(\xi^2+\rho_1^2)^{\frac{1}{2}}z} - (\xi^2+\rho_2^2)^{\frac{1}{2}} e^{-(\xi^2+\rho_2^2)^{\frac{1}{2}}z}] \cos(\xi x)d\xi$$

$$= \frac{z^2}{r^2}[\rho_2^2 K_2(\rho_2 r) - \rho_1^2 K_2(\rho_1 r)] + \frac{y^2}{r^3}[\rho_2 K_1(\rho_2 r) - \rho_1 K_1(\rho_1 r)],$$

where $r^2 = x^2 + z^2$. The remaining integrals in equations (13.3) and (13.4) are in a form suitable for treatment by a method similar to that used by Lamb (1904).

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XI.—The Free Commutative Entropic Logarithmetic.*

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SYNOPSIS

The commutative and entropic congruence relations determine a homomorphism on the free logarithmetic $\mathfrak{L}$, the arithmetic of the indices of powers of the generating element of a free cyclic groupoid. A necessary and sufficient condition that two indices should be concordant (*i.e.* congruent in the free commutative entropic logarithmetic) is that the bifurcating trees corresponding to these indices should have the same number of free ends at each altitude. It follows that the free commutative entropic logarithmetic can be represented faithfully by index ψ-polynomials (or θ-polynomials) in one indeterminate.

In the concluding section enumeration formulæ are obtained for the number of non-concordant indices of a given altitude and for the number of indices concordant to a given index.

1. INTRODUCTION

THE present paper is a continuation of a previous paper (Minc 1957 ‡) and uses the nomenclature of the latter without further explanation.

Addition in the free logarithmetic $\mathfrak{L}$ (*ibid.*, p. 321) is non-associative and non-commutative. Multiplication in $\mathfrak{L}$ is associative and right-distributive but not commutative nor left-distributive. The following congruence relations determine therefore homorphisms on $\mathfrak{L}$:

commutative: $$P+Q \sim Q+P, \qquad (c)$$

palintropic: $$PQ \sim QP, \qquad (p)$$

left-distributive: $$(P+Q)R \sim PR+QR, \qquad (d)$$

and entropic: $$(P+Q)+(R+S) \sim (P+R)+(Q+S) \qquad (e)$$

(*cf.* Etherington 1949).

We denote the homomorph of $\mathfrak{L}$ determined by congruence relation (r) by $\mathfrak{L}_r$. It is known that $\mathfrak{L}_e$ is a homomorph of $\mathfrak{L}_p$ and that $\mathfrak{L}_p$ is isomorphic

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‡ I take this opportunity to correct a mis-statement on p. 335 of that paper. In line 21 *for* "the potency of a prime tree is a prime number" *read* "a tree is prime if its potency is a prime number".

to $\mathfrak{L}_d$. The above four relations determine therefore only five distinct homomorphs: the free commutative logarithmetic $\mathfrak{L}_c$, the free palintropic logarithmetic $\mathfrak{L}_p$, the free entropic logarithmetic $\mathfrak{L}_e$, the free commutative palintropic logarithmetic $\mathfrak{L}_{cp}$ and the free commutative entropic logarithmetic $\mathfrak{L}_{ce}$. The study of $\mathfrak{L}_{ce}$ and its representations by ψ- and θ-polynomials in one indeterminate forms the main part of this paper. The paper concludes with a section on enumeration of indices.

The author is indebted to Dr I. M. H. Etherington for many helpful suggestions.

2. Congruence Relations on $\mathfrak{L}$

$P=Q$ means that P and Q represent the same index in $\mathfrak{L}$ or the same tree in $\mathfrak{T}$ (*v.* Minc 1957, p. 322). We shall say that P is *congruent* to Q modulo (r) and write $P \sim Q \bmod (r)$ if (r) is an equivalence relation on $\mathfrak{L}$ and

either (i) $P=Q$;

or (ii) $P \sim Q \bmod (r)$ by *direct* application of (r) (*e.g.* $2+3 \sim 3+2 \bmod (c)$; $2 \cdot 3 \sim 3 \cdot 2 \bmod (p)$; *etc.*);

or (iii) $P=P'+P''$, $Q=Q'+Q''$ and $P' \sim Q'$, $P'' \sim Q'' \bmod (r)$;

or (iii′) $P=RP'$ and $Q=RQ'$ and $P' \sim Q' \bmod (r)$;

or (iv) $P=R_1 \sim R_2 \sim \ldots \sim R_k=Q$ where $R_i \sim R_{i+1} \bmod (r)$ $(1 \leqslant i \leqslant k-1)$ by virtue of (i) or (ii) or (iii) or (iii′).

We prove that "congruence" on $\mathfrak{L}$ thus defined is a congruence relation in the usual sense (*cf.* Birkhoff 1948, p. vii). It is obviously a congruence relation for addition. It suffices to prove

Theorem 1.—If $P \sim P' \bmod (r)$ and $Q \sim Q' \bmod (r)$ then $PQ \sim P'Q' \bmod (r)$.

Proof.—$PQ \sim PQ'$, by (iii′). We prove that the premises of the theorem imply $PQ' \sim P'Q' \bmod (r)$. Use non-associative induction on Q' (*v.* Minc 1957, p. 322, footnote). If $Q'=1$ there is nothing to prove. Otherwise let $Q'=Q_1'+Q_2'$. Assume that $PQ_1' \sim P'Q_1'$ and $PQ_2' \sim P'Q_2'$. Then, by (iii), $PQ_1'+PQ_2' \sim P'Q_1'+P'Q_2'$, *i.e.* $PQ' \sim P'Q'$, since multiplication in $\mathfrak{L}$ is right-distributive.

For all congruence relations considered in the present paper case (iii′) of the definition follows from the other four cases.

Theorem 2.—Let ρ_1, ρ_2, ρ_3 be equivalence relations on $\mathfrak{L}$ defined as follows:

(1) $P \rho_1 Q$ if (I) $P=S+T$ and $Q=T+S$;

(2) $P\rho_2 Q$ if (I) $P=ST$ and $Q=TS$;

(3) $P\rho_3 Q$ if (I) $P=(S+T)+(U+V)$ and $Q=(S+U)+(T+V)$;

also $P\rho_i Q$ $(i=1, 2, 3)$ if

either (II) $P=Q$;

or (III) $P=P'+P''$, $Q=Q'+Q''$ where $P'\rho_i Q'$ and $P''\rho_i Q''$;

or (IV) $P=R_1\rho_i R_2\rho_i R_3\rho_i \ldots \rho_i R_k=Q$ where $R_s\rho_i R_{s+1}$ by virtue of (I) or (II) or (III).

Then $P\rho_1 Q$ is equivalent to $P\sim Q \bmod (c)$,

$P\rho_2 Q$ is equivalent to $P\sim Q \bmod (p)$

and $P\rho_3 Q$ is equivalent to $P\sim Q \bmod (e)$.

Proof.—ρ_1, ρ_2, ρ_3 are obviously congruence relations for addition. It remains to prove that $P\rho_i Q$ implies $RP\rho_i RQ$ $(i=1, 2, 3)$. We consider the four cases in which $P\rho_i Q$.

(1) If (I) $P=S+T$ and $Q=T+S$ then

$R(S+T)=RS+RT$, since multiplication in $\mathfrak{L}$ is right-distributive,

$\rho_1 RT+RS$, by (ρ_1),

$=R(T+S)$.

Hence, by (IV), $RP\rho_1 RQ$.

(2) If (I) $P=ST$ and $Q=TS$ the proof is by non-associative induction.

When $T=1$, $P=Q$ and thus $RP\rho_2 RQ$, by (II). Let $T=T'+T''$ and assume that $RST'\rho_2 RT'S$ and $RST''\rho_2 RT''S$.

Then $RST=RST'+RST''$

$\rho_2 RT'S+RT''S$, by the induction hypothesis and (III),

$\rho_2 SRT'+SRT''$, since $(RT')S\rho_2 S(RT')$ and $(RT'')S\rho_2 S(RT'')$,

$=SR(T'+T'')$

$=SRT=RTS$, by (ρ_2).

Hence $RP\rho_2 RQ$.

(3) If $P=(S+T)+(U+V)$, $Q=(S+U)+(T+V)$ then

$RP=(RS+RT)+(RU+RV)$, since multiplication in $\mathfrak{L}$ is right-distributive,

$\rho_3(RS+RU)+(RT+RV)$, by (ρ_3),

$=R\{(S+U)+(T+V)\}$

$=RQ$.

Hence $RP\rho_3 RQ$.

Further, for all three relations:

if (II) $P=Q$, we have $RP=RQ$ and thus $RP\rho_i RQ$;

if (III) $P=P'+P''$, $Q=Q'+Q''$, where $P'\rho_i Q'$, $P''\rho_i Q''$, the result is easily provable by induction on altitude (or potency) of P;

if (IV) $P=R_1\rho_i R_2\rho_i R_3\rho_i \ldots \rho_i R_k=Q$, where $R_s\rho_i R_{s+1}$ by virtue of (I), (II) or (III), the proof is by induction on k.

It follows from the above theorem that congruence relations mod (ϕ), (c), (e) are completely defined by cases (i), (ii), (iii) and (iv) of the definition in § 2 and in all subsequent proofs in which it is premised that indices or trees are congruent it will suffice to consider these four cases only.

3. Free Commutative Logarithmetic. Free Entropic Logarithmetic

Commutative logarithmetics have been studied by Etherington (1939, 1940 and 1949). In this paper we only add a theorem on faithful representations of the free commutative logarithmetic $\mathfrak{L}_c$.

Denote the homomorphs of Ψ, Θ, Ω (the algebras of all ψ-, θ-, ω-polynomials) determined by congruence relations

$$\psi(\lambda, \mu)\sim\psi(\mu, \lambda), \qquad \theta(\lambda, \mu)\sim\theta(\mu, \lambda), \qquad \omega(\lambda, \mu)\sim\omega(\mu, \lambda)$$

by Ψ_c, Θ_c, Ω_c respectively.

Theorem 3.—$\mathfrak{L}_c$ is faithfully represented by Ψ_c; also by Θ_c and by Ω_c.

Proof.—We prove first that

$$\psi(\lambda, \mu)\sim\psi(\mu, \lambda) \tag{c}$$

implies $\psi_{P+Q}\sim\psi_{Q+P}$.

$$\begin{aligned}
\psi_{P+Q}(\lambda, \mu) &= \psi_P(\lambda, \mu).\lambda+\psi_Q(\lambda, \mu).\mu\\
&\sim \psi_{P+Q}(\mu, \lambda), \qquad \text{by (c)},\\
&= \psi_P(\mu, \lambda).\mu+\psi_Q(\mu, \lambda).\lambda\\
&\sim \psi_P(\lambda, \mu).\mu+\psi_Q(\lambda, \mu).\lambda, \qquad \text{by (c)},\\
&= \psi_{Q+P}(\lambda, \mu).
\end{aligned}$$

To prove the converse, *i.e.* that $\psi_{P+Q}\sim\psi_{Q+P}$ implies (c), note that $\psi_1\sim\psi_1$ mod (c) and use induction on the altitude of $P+Q$. When $\alpha_{P+Q}=1$,

$\psi_{P+Q}(\lambda, \mu) = \lambda + \mu$ while $\psi_{P+Q}(\mu, \lambda) = \mu + \lambda$. Suppose the theorem holds for altitudes less than a ($a > 1$) and let $\alpha_{P+Q} = a$. Then

$$\begin{aligned}\psi_{P+Q}(\lambda, \mu) &\sim \psi_{Q+P}(\lambda, \mu)\\ &= \psi_P(\lambda, \mu).\mu + \psi_Q(\lambda, \mu).\lambda\\ &\sim \psi_P(\mu, \lambda).\mu + \psi_Q(\mu, \lambda).\lambda, \quad \text{by the induction hypothesis since } \alpha_P, \alpha_Q < a,\\ &= \psi_{P+Q}(\mu, \lambda).\end{aligned}$$

The proof for θ- and ω-polynomials is almost identical.

The free entropic logarithmetic $\mathfrak{L}_e$ is a homomorph of the free palintropic logarithmetic, that is to say PQ and QP are congruent in $\mathfrak{L}_e$ for all P and Q. This result has been essentially obtained by Murdoch (1939, Corollary to Theorem 10) and in a more general form by Etherington (1949, Theorem 4). Etherington has also proposed the question (1951, p. 249) whether the free entropic logarithmetic is represented faithfully by index θ-polynomials in *commuting* indeterminates λ, μ. Call index polynomials in commuting indeterminates *palindromic*. It is known that palindromic θ-polynomials represent faithfully the logarithmetic of the general train algebra of rank 3 (*v.* Etherington 1951, p. 249). Etherington's question amounted therefore to this: are the free entropic logarithmetic and the logarithmetic of the general train algebra of rank 3 isomorphic? In 1954 I communicated to Dr Etherington the following example which answers the question in the negative.

Example.—The indices $(4+1)+(1+3)$ and $(3+1)+(1+4)$ are not congruent mod (e) although their palindromic θ-polynomials are both equal to $\lambda^2\mu^2 + \lambda^2\mu + \lambda\mu^2 + \lambda^2 + \mu^2 + \lambda + \mu + 1$.

4. Free Commutative Entropic Logarithmetic

This logarithmetic has particularly interesting faithful representations by index polynomials. We introduce for it a special nomenclature and notation. If two indices or trees, P and Q, are congruent mod (c) (e) we call them *concordant* and we write $P \backsim Q$.

Observe that $(P+Q)+(R+S) \backsim (A+B)+(C+D)$ where (A, B, C, D) is any of the 4! permutations of (P, Q, R, S). Indeed this fact together with the relation $1+P \backsim P+1$ are equivalent to (c) (e). This suggests

Theorem 4.—If two subtrees of P of the same order * be transposed the resulting tree Q is concordant to P.

* *V.* Minc 1957, p. 322.

Proof.—The theorem holds trivially when P is of altitude 1. We use induction on altitude and assume that the theorem is true for trees of altitudes less than a. Let P be of altitude a.

(1) If $P=P'+1$ or $1+P'$, both subtrees must belong to P', a tree of altitude $a-1$; the result follows by the induction hypothesis.

(2) If $P=(P_1+P_2)+(P_3+P_4)$ then:

(a) If both subtrees belong to P_1+P_2, a tree of altitude $a-1$, the theorem again follows by the induction hypothesis. Similarly if the two subtrees belong to P_3+P_4.

(b) If one subtree belongs to P_1 and the other to P_3 (or one to P_2 and the other to P_4) the result follows from (a) since

$$(P_1+P_2)+(P_3+P_4) \sim (P_1+P_3)+(P_2+P_4).$$

(c) If one subtree belongs to P_1 and the other to P_4 (or one to P_2 and the other to P_3) the result follows from (a) since

$$(P_1+P_2)+(P_3+P_4) \sim (P_1+P_2)+(P_4+P_3) \sim (P_1+P_4)+(P_2+P_3).$$

(*Note.*—The proof of the equivalent proposition (which is false; *v.* Example in § 3) for the non-commutative entropic logarithmetic fails in case (2) (c).)

We shall require a more general form of this result.

LEMMA.—If P has a free end at altitude a then any tree Q concordant to it has also a free end at the same altitude.

Proof.—The lemma is quite obvious

if (i) $P=Q$,

or if (ii) (1) $P=(R+S)+(T+U)$ and $Q=(R+T)+(S+U)$

or if (ii) (2) $P=R+S$ and $Q=S+R$.

It is easily provable by induction on altitude

if (iii) $P=R+S$, $Q=R'+S'$ and $R \sim R'$, $S \sim S'$

and by induction on k

if (iv) $P=R_1 \sim R_2 \sim \ldots \sim R_k=Q$.

If R is a subordinate of the nth order of P (*v.* Minc 1957, p. 325) we shall call P a *superior* of the nth order of R. We shall use the following notation: Let a superior of the first order of R be denoted by $\bar{R}$ or by $\bar{R}_{(\alpha)}$ if the node of the additional fork in the superior is at altitude α. $\bar{R}$ (or $\bar{R}_{(\alpha)}$) denotes a definite though unspecified tree, not the set of all trees having R for a subordinate.

THEOREM 5.—If P and Q are two concordant trees each with a free end at altitude a then $\bar{P}_{(a)} \sim \bar{Q}_{(a)}$.

Proof.—Consider in turn the four cases defining $P \sim Q$.

(i) If $P=Q$ the result follows from Theorem 4.

(ii) (1) $P=(R+S)+(T+U)$ and $Q=(R+T)+(S+U)$. One at least of R, S, T, U has a free end at altitude $a-2$; let it be R. Then

$$\begin{aligned}\bar{P}_{(a)} &\sim (\bar{R}_{(a-2)}+S)+(T+U), \quad \text{by Theorem 4,}\\ &\sim (\bar{R}_{(a-2)}+T)+(S+U), \quad \text{by } (e),\\ &\sim \bar{Q}_{(a)}, \quad \text{by Theorem 4.}\end{aligned}$$

(ii) (2) $P=R+S$ and $Q=S+R$. First suppose that R has a free end at altitude $a-1$. Then

$$\begin{aligned}\bar{P}_{(a)} &\sim \bar{R}_{(a-1)}+S, \quad \text{by Theorem 4,}\\ &\sim S+\bar{R}_{(a-1)}, \quad \text{by } (c),\\ &\sim \bar{Q}_{(a)}, \quad \text{by Theorem 4.}\end{aligned}$$

If R has no free ends at altitude $a-1$, S must have one; the proof is then similar.

(iii) $P=R+S$, $Q=R'+S'$ and $R \sim R'$, $S \sim S'$. Suppose that R has a free end at altitude $a-1$. Then, by the lemma, R' has a free end at the same altitude and $\bar{P}_{(a)} \sim \bar{R}_{(a-1)}+S$, $\bar{Q}_{(a)} \sim \bar{R}'_{(a-1)}+S'$. These are concordant if $\bar{R}_{(a-1)} \sim \bar{R}'_{(a-1)}$. Use therefore induction on a. Again, if R has no free ends at altitude $a-1$ then S must have one and the proof is similar.

(iv) If $P=R_1 \sim R_2 \sim \ldots \sim R_k=Q$ then by the lemma each R_i has a free end at altitude a. The proof is by induction on k.

We are now in a position to prove the principal theorem on the structure of concordant trees.

THEOREM 6.—Two trees are concordant if and only if they have the same number of free ends at each altitude.

Proof.—Let the two trees be P and Q.

Necessity.—Let p_i, q_i, r_i, s_i, t_i, u_i denote the numbers of free ends at altitude i in trees P, Q, R, S, T, U respectively.

(i) If $P=Q$ there is nothing to prove.

(ii) (1) If $P=(R+S)+(T+U)$ and $Q=(R+T)+(S+U)$ then $p_i=q_i=r_{i-2}+s_{i-2}+t_{i-2}+u_{i-2}$ $(2 \leqslant i \leqslant \alpha_P)$ and $p_0=p_1=q_0=q_1=0$.

(ii) (2) If $P=R+S$ and $Q=S+R$ then $p_i=q_i=r_{i-1}+s_{i-1}$ $(1 \leqslant i \leqslant \alpha_P)$ and $p_0=q_0=0$.

(iii) $P=R+S$, $Q=T+U$ and $R \sim T$, $S \sim U$. The condition is obviously necessary if $\alpha_P=1$. Use induction on altitude of P. The lemma to Theorem 5 implies that altitudes of concordant trees are equal. The altitudes of P and Q are therefore equal and those of R, S, T, U are all less than α_P. Thus, by the induction hypothesis, $r_i=t_i$ and $s_i=u_i$ for all i. But $p_i=r_{i-1}+s_{i-1}$, $q_i=t_{i-1}+u_{i-1}$. Hence $p_i=q_i$.

(iv) If $P=R_1 \sim R_2 \sim \ldots \sim R_k=Q$ the necessity is proved by induction on k.

Sufficiency.—Note that the potency of any tree T is $\delta_T=\sum_i t_i$ and use induction on the potency of P.

If $\delta_P=1$, $P=Q=1$ and the condition is obviously sufficient. Assume that it is sufficient for trees of potency less than d. Let $\delta_P=\delta_Q=d$, $\alpha_P=\alpha_Q=a$ and let R, S be the first principal subordinates (*v.* Minc 1957, p. 325) of P, Q respectively. Then, since $p_i=q_i$ $(1 \leqslant i \leqslant a)$, $r_i=s_i=p_i$ $(1 \leqslant i \leqslant a-2)$, $r_{a-1}=s_{a-1}=p_{a-1}+1$ and $r_a=s_a=p_a-2$. But the potencies of R and S are equal to $d-1$. Thus, by the induction hypothesis, R and S are concordant and, by Theorem 5, $\bar{R}_{(a-1)} \sim \bar{S}_{(a-1)}$. Now, by Theorem 4, $P \sim \bar{R}_{(a-1)}$ and $Q \sim \bar{S}_{(a-1)}$. Hence the result.

A similar necessary and sufficient condition can be obtained for numbers of nodes (or of all knots) at each altitude.

5. ψ- AND θ-POLYNOMIALS IN ONE INDETERMINATE

The altitude of a knot is equal to the degree in λ, μ of its term (*v.* Minc 1957). This and Theorem 6 suggest that concordant trees (or indices) can be represented by polynomials in one indeterminate in which the degree of each term corresponds to the altitude and the coefficient to the number of free ends at this altitude. We now introduce such index polynomials, study their properties and prove that $\mathfrak{L}_{ce}$ is faithfully represented by them. It turns out that these are Etherington's original index polynomials (*cf.* Etherington 1940).

The algebras of the two types of index polynomials (*cf.* Minc 1957) defined below are homomorphs of Ψ and Θ determined by the congruence relations:

$$\psi(\lambda, \mu) \sim \psi'(\lambda, \mu) \quad \text{if} \quad \psi(\lambda, \lambda)=\psi'(\lambda, \lambda) \quad \text{in} \quad \mathfrak{M}[\lambda, \mu];$$
$$\theta(\lambda, \mu) \sim \theta'(\lambda, \mu) \quad \text{if} \quad \theta(\lambda, \lambda)=\theta'(\lambda, \lambda) \quad \text{in} \quad \mathfrak{M}[\lambda, \mu].$$

It is convenient therefore to call them *index ψ- and θ-polynomials in one indeterminate* and to denote them by $\psi_P(\lambda)$ and $\theta_P(\lambda)$ or, where no confusion is likely to arise (as in this section), simply ψ- and θ-polynomials and to write ψ_P and θ_P.

Definitions.—(i) Index ψ-polynomials in one indeterminate:

$$\psi_1(\lambda)=1, \qquad \psi_{P+Q}(\lambda)=\lambda\{\psi_P(\lambda)+\psi_Q(\lambda)\}.$$

(ii) Index θ-polynomials in one indeterminate:

$$\theta_1(\lambda)=0, \qquad \theta_{P+Q}(\lambda)=\lambda\{\theta_P(\lambda)+\theta_Q(\lambda)\}+1.$$

We have $\psi_{I'}=(2\lambda-1)\theta_P+1$. This is easily proved by non-associative induction. For, since $\theta_1=0$, $\psi_1=(2\lambda-1)\theta_1+1$ and if we assume that $\psi_Q=(2\lambda-1)\theta_Q+1$ and $\psi_R=(2\lambda-1)\theta_R+1$ then

$$\begin{aligned}\psi_{Q+R}&=\lambda(\psi_Q+\psi_R)\\&=\lambda\{(2\lambda-1)\theta_Q+1+(2\lambda-1)\theta_R+1\}\\&=(2\lambda-1)\{\lambda(\theta_Q+\theta_R)+1\}+1\\&=(2\lambda-1)\theta_{Q+R}+1.\end{aligned}$$

Call the term of maximal degree in λ in a polynomial $\phi(\lambda)$ the *leading* term of $\phi(\lambda)$. It is easily seen that all coefficients in index polynomials defined above are non-negative integers and that the coefficient of the leading term of ψ_P ($P\neq 1$) is even.

THEOREM 7.—The polynomial $\phi=2\lambda^i+\phi'$, where ϕ' is a polynomial in λ with non-negative integer coefficients, is an index ψ-polynomial if and only if $\lambda^{i-1}+\phi'$ is one.

Proof.—If $2\lambda^i+\phi'$ is an index ψ-polynomial, ψ_P say, then $(2\lambda^i+\phi')-2\lambda^i+\lambda^{i-1}=\lambda^{i-1}+\phi'$ is the ψ-polynomial of a first subordinate of either P or of a tree concordant to P. Again, if $\lambda^{i-1}+\phi'$ is a ψ-polynomial, ψ_Q say, then $(\lambda^{i-1}+\phi')-\lambda^{i-1}+2\lambda^i=2\lambda^i+\phi'$ is the ψ-polynomial of $\overline{Q}_{(i-1)}$, a first superior of Q. To these somewhat loose remarks we add a formal proof.

Necessity.—Use induction on $n(\phi)$, the degree of ϕ. If $n(\phi)=1$, $\phi=2\lambda$, *i.e.* $i=1$, $\phi'=0$ and therefore $\lambda^{i-1}+\phi'=1=\psi_1$. Assume that the condition is necessary for ψ-polynomials of degree less than m. Let $n(\phi)=m$. Then $\phi=2\lambda^i+\phi'=\lambda\psi_A+\lambda\psi_B$ and either (*a*) ψ_A or ψ_B contains a term $\kappa\lambda^{i-1}$ with $\kappa\geqslant 2$; or (*b*) ψ_A and ψ_B each contains a term λ^{i-1}.

If (*a*): suppose that ψ_B contains a term $\kappa\lambda^{i-1}$ and let $\psi_B=2\lambda^{i-1}+\phi_B$ where ϕ_B is a polynomial with non-negative coefficients. Then, since $n(\psi_B)\leqslant m-1$, $\lambda^{i-2}+\phi_B$ is a ψ-polynomial and

$$\begin{aligned}\lambda^{i-1}+\phi' &= \phi - 2\lambda^i + \lambda^{i-1}\\ &= \lambda\psi_A + \lambda\psi_B - 2\lambda^i + \lambda^{i-1}\\ &= \lambda\psi_A + (2\lambda^i + \lambda\phi_B) - 2\lambda^i + \lambda^{i-1}\\ &= \lambda\psi_A + \lambda(\lambda^{i-2} + \phi_B)\end{aligned}$$

is also a ψ-polynomial.

If (δ): let $\psi_A = \lambda\psi_{A_1} + \lambda\psi_{A_2}$, $\psi_B = \lambda\psi_{B_1} + \lambda\psi_{B_2}$ and suppose that ψ_{A_1} and ψ_{B_1} each contains a term λ^{i-2}. Then $\phi = \lambda(\lambda\psi_{A_1} + \lambda\psi_{B_1}) + \lambda(\lambda\psi_{A_2} + \lambda\psi_{B_2}) = \lambda\psi_C + \lambda\psi_D$, say, where ψ_C contains the term $2\lambda^{i-2}$ and the proof proceeds as in case (a).

Sufficiency.—Let $\psi_P = \lambda^{i-1} + \phi'$. Then $\phi = \psi_P + (2\lambda - 1)\lambda^{i-1}$. Use induction on $n(\psi_P)$. If $n(\psi_P) = 0$ $\psi_P = 1$, $i = 1$ and $\phi = 1 + (2\lambda - 1) = 2\lambda = \psi_2$. Suppose that the condition is sufficient for polynomials of degree less than m $(m > 0)$ and let $n(\psi_P) = m$. Then $\psi_P = \lambda\psi_Q + \lambda\psi_R$ and either ψ_Q or ψ_R contains a term $\kappa\lambda^{i-2}(\kappa > 0)$; let it be ψ_R. Hence ψ_R can be written in the form $\lambda^{i-2} + \phi_R$, where ϕ_R is a polynomial with non-negative coefficients, and, since $n(\psi_R) < m$, $2\lambda^{i-1} + \phi_R$ is an index ψ-polynomial. But

$$\begin{aligned}\phi &= \psi_P + (2\lambda - 1)\lambda^{i-1}\\ &= \lambda\psi_Q + \lambda(\lambda^{i-2} + \phi_R) + (2\lambda - 1)\lambda^{i-1}\\ &= \lambda\psi_Q + \lambda(\phi_R + 2\lambda^{i-1})\end{aligned}$$

and is therefore also a ψ-polynomial.

COROLLARY 1.—A necessary and sufficient condition for $\phi(\lambda)$, a polynomial of degree n $(n > 1)$ with positive integer coefficients to be an index ψ-polynomial is that $\phi - (2\lambda - 1)\lambda^{n-1}$ should be one.

COROLLARY 2.—$\phi(\lambda)$, a polynomial with positive coefficients and leading term $\kappa\lambda^n$, is an index ψ-polynomial if and only if $\phi - (2\lambda - 1)\frac{1}{2}\kappa\lambda^{n-1}$ is one.

Necessary and sufficient conditions that $\sum_{i=0}^{n} \kappa_i\lambda^i$ should be an index θ-polynomial have been given by Etherington (1951, p. 251). They are: (i) all κ_i are non-negative integers; (ii) if $n \neq 0$, $\kappa_i < 2\kappa_{i-1}$; (iii) $\kappa_0 = 0$ or 1. The necessity of these is quite obvious. The sufficiency can be proved by above Corollary 2. For $\sum_{i=0}^{n} \kappa_i\lambda^i$ is a θ-polynomial if $\phi = (2\lambda - 1)(\sum_{i=0}^{n} \kappa_i\lambda^i) + 1$ is a ψ-polynomial and this is so if

$$\phi' = (2\lambda - 1)(\sum_{i=0} \kappa_i\lambda^i) + 1 - (2\lambda - 1)\kappa_n\lambda^n = (2\lambda - 1)(\sum_{i=0}^{n-1} \kappa_i\lambda^i) + 1$$

is one. Now the degree of ϕ' is less than that of ϕ. The proof is by induction on degree.

It is worth noting that if $\sum_{i=0}^{n} \kappa_i \lambda^i$ is an index θ-polynomial then so is $\sum_{i=0}^{r} \kappa_i \lambda^i$ $(0 \leqslant r \leqslant n)$.

THEOREM 8.—The free commutative entropic logarithmetic is faithfully represented by index ψ-polynomials in one indeterminate.

Proof. I.—To prove that concordant indices have the same ψ-polynomial, *i.e.* that $(P \sim Q) \Longrightarrow \{\psi_P(\lambda) = \psi_Q(\lambda)\}$:

(i) If $P = Q$ then obviously $\psi_P = \psi_Q$.

(ii) (1) If $P = (R+S)+(T+U)$ and $Q = (R+T)+(S+U)$ then

$$\psi_P = \psi_Q = \lambda^2(\psi_R + \psi_S + \psi_T + \psi_U).$$

(ii) (2) If $P = R+S$ and $Q = S+R$ then

$$\psi_P = \psi_Q = \lambda(\psi_R + \psi_S).$$

(iii) If $P = R+S$, $Q = T+U$ and $R \sim T$, $S \sim U$ then $\psi_P = \lambda(\psi_R + \psi_S)$ and $\psi_Q = \lambda(\psi_T + \psi_U)$ which are equal if $\psi_R = \psi_T$ and $\psi_S = \psi_U$. Use induction on altitude.

(iv) If $P = R_1 \sim R_2 \sim \ldots \sim R_k = Q$, use induction on k.

II.—To prove that $(\psi_P = \psi_Q) \Longrightarrow (P \sim Q)$: $\psi_P = \psi_Q$ implies that $\delta_P = \delta_Q$ and $\alpha_P = \alpha_Q$. Use induction on δ_P. If $\delta_P = 1$, $P = Q = 1$. Assume that the theorem holds for indices of potency less than d. Let $\delta_P = d$. Consider the trees P and Q. Their first principal subordinates have both the ψ-polynomial $\psi_P - (2\lambda - 1)\lambda^{\alpha_P - 1}$. The potency of these subordinates is $d-1$ so that, by the induction hypothesis, they are concordant. Now, P and Q are their superiors satisfying the premises of Theorem 5. The result follows.

COROLLARY.—The free commutative entropic logarithmetic is faithfully represented by index θ-polynomials in one indeterminate.

For

$$\begin{aligned}(P \sim Q) &\Longleftrightarrow (\psi_P = \psi_Q) \\ &\Longleftrightarrow \{(2\lambda - 1)\theta_P + 1 = (2\lambda - 1)\theta_Q + 1\} \\ &\Longleftrightarrow (\theta_P = \theta_Q).\end{aligned}$$

6. ENUMERATION OF INDICES

The numbers a_δ, p_α of possible indices in $\mathfrak{L}$ of given potency δ and of given altitude α respectively are given (*v.* Etherington 1939) by the recurrence formulæ

$$a_\delta = a_1 a_{\delta-1} + a_2 a_{\delta-2} + a_3 a_{\delta-3} + \ldots + a_{\delta-1} a_1, \qquad a_1 = 1;$$
$$p_{\alpha+1} = 2p_\alpha(p_0 + p_1 + p_2 + \ldots + p_{\alpha-1}) + p_\alpha^2, \qquad p_0 = 1.$$

The formulæ for b_δ, q_α, the corresponding numbers of possible non-congruent indices in $\mathfrak{Q}_c$, are (*ibid.*):

$$b_1 = b_2 = q_0 = 1,$$
$$b_{2\delta-1} = b_1 b_{2\delta-2} + b_2 b_{2\delta-3} + \ldots + b_{\delta-1} b_\delta,$$
$$b_{2\delta} = b_1 b_{2\delta-1} + b_2 b_{2\delta-2} + \ldots + b_{\delta-1} b_{\delta+1} + \tfrac{1}{2} b_\delta (b_\delta + 1);$$
$$q_{\alpha+1} = q_\alpha(q_0 + q_1 + q_2 + \ldots + q_{\alpha-1}) + \tfrac{1}{2} q_\alpha (q_\alpha + 1).$$

If we denote the number of indices in $\mathfrak{Q}$ of altitude not greater than α by s_α, *i.e.* $s_\alpha = \sum_{i=0}^{\alpha} p_i$, the formula for $p_{\alpha+1}$ becomes

$$\begin{aligned} p_{\alpha+1} &= 2p_\alpha s_{\alpha-1} + p_\alpha^2 \\ &= (s_{\alpha-1} + p_\alpha)^2 - s_{\alpha-1}^2 \\ &= s_\alpha^2 - s_{\alpha-1}^2. \end{aligned}$$

Alternatively,

$$\begin{aligned} p_{\alpha+1} &= p_\alpha(s_\alpha + s_{\alpha-1}) \\ &= \prod_{i=1}^{\alpha} (s_i + s_{i-1}). \end{aligned}$$

The first of these formulæ gives

$$s_{\alpha+1} = 1 + s_\alpha^2.$$

Similarly if $t_\alpha = \sum_{i=0}^{\alpha} q_i$ we obtain the corresponding formulæ

$$2q_{\alpha+1} = t_\alpha^2 - t_{\alpha-1}^2 + q_\alpha = q_\alpha(t_\alpha + t_{\alpha-1} + 1),$$
$$q_{\alpha+1} = 2^{-\alpha} \prod_{i=1}^{\alpha} (t_i + t_{i-1} + 1),$$
$$2t_{\alpha+1} = t_\alpha^2 + t_\alpha + 2.$$

The enumeration of indices in other free logarithmetics discussed in this paper is more difficult. I give below a formula for r_α, the number of all non-concordant indices of a given altitude. As far as I am aware, the other relevant enumeration formulæ have not yet been found.

Let κ be a non-negative integer, λ an indeterminate and i any non-negative integer such that $2^i \geqslant \kappa$. Denote by Λ the operator defined as follows:

$$\Lambda(\kappa\lambda^i) = \kappa\lambda^i \quad \text{if} \quad \kappa = 0 \text{ or } 1$$

and

$$\Lambda(\kappa\lambda^i) = (\kappa - 2)\lambda^i + \lambda^{i-1} \quad \text{if} \quad \kappa \geqslant 2.$$

Define the Λ-value of κ, denoted Λ_κ, as the number of all possible (different) polynomials in λ obtained by operating with Λ in all possible manners on $\kappa\lambda^i$ and on terms of thus derived polynomials.

Example.—To find the Λ-value of 7. We have

$$\begin{aligned} 7\lambda^i, \\ \Lambda(7\lambda^i) &= 5\lambda^i + \lambda^{i-1}, \\ (\Lambda(5\lambda^i)) + \lambda^{i-1} &= 3\lambda^i + 2\lambda^{i-1}, \\ (\Lambda(3\lambda^i)) + 2\lambda^{i-1} &= \lambda^i + 3\lambda^{i-1}, \\ 3\lambda^i + \Lambda(2\lambda^{i-1}) &= 3\lambda^i + \lambda^{i-2}, \\ \lambda^i + \Lambda(3\lambda^{i-1}) &= \lambda^i + \lambda^{i-1} + \lambda^{i-2}, \end{aligned}$$

i.e. 6 distinct polynomials and it is impossible to obtain more than 6. Hence the Λ-value of 7 is 6.

It is easily seen that

$$\begin{aligned} &\Lambda_0 = 1, \quad \Lambda_4 = 4, \quad \Lambda_8 = 10, \quad \Lambda_{12} = 20, \\ &\Lambda_1 = 1, \quad \Lambda_5 = 4, \quad \Lambda_9 = 10, \quad \Lambda_{13} = 20, \\ &\Lambda_2 = 2, \quad \Lambda_6 = 6, \quad \Lambda_{10} = 14, \quad \Lambda_{14} = 26, \\ &\Lambda_3 = 2, \quad \Lambda_7 = 6, \quad \Lambda_{11} = 14, \quad \Lambda_{15} = 26, \text{ etc.} \end{aligned}$$

In fact we have

LEMMA.—

$$\Lambda_{2\kappa+1} = \Lambda_{2\kappa} = \sum_{r=0}^{\kappa} \Lambda_r.$$

Proof.—Use induction on κ. The formula gives correct Λ-value for $\kappa = 1$. Assume that the formula holds for integers less than κ. Consider $\Lambda_{2\kappa}$, the number of all distinct polynomials which can be obtained from $2\kappa\lambda^i$ by the process described above. All such polynomials with a term in λ^i are obtained from $2\kappa\lambda^i = 2\lambda^i + 2(\kappa-1)\lambda^i$ by operating with Λ in all possible manners on the term $2(\kappa-1)\lambda^i$ and on terms derived from it. There are $\Lambda_{2(\kappa-1)}$ such polynomials. All the derived polynomials of degree less than i are obtained by operating in the same way on $\kappa\lambda^{i-1}$. There are Λ_κ of these. Hence $\Lambda_{2\kappa} = \Lambda_{2(\kappa-1)} + \Lambda_\kappa$. But, by the induction hypothesis, $\Lambda_{2(\kappa-1)} = \sum_{r=0}^{\kappa-1} \Lambda_r$. Thus $\Lambda_{2\kappa} = \sum_{r=0}^{\kappa-1} \Lambda_r + \Lambda_\kappa = \sum_{r=0}^{\kappa} \Lambda_r$. $\Lambda_{2\kappa+1}$ is the number of possible polynomials obtained from $(2\kappa+1)\lambda^i$ by the same process. Now, $(2\kappa+1)\lambda^i = \lambda^i + 2\kappa\lambda^i$ and since $\Lambda(\lambda^i) = \lambda^i$ all the required polynomials are obtained by operating with Λ on the term $2\kappa\lambda^i$ and on terms derived from it. Thus $\Lambda_{2\kappa+1} = \Lambda_{2\kappa}$.

Denote by r_α the number of all index ψ-polynomials, $\psi(\lambda)$, of degree α, *i.e.* the number of all non-concordant indices of altitude α.

THEOREM 9.

$$r_{\alpha+1}=\sum_{i=0}^{2^\alpha-1}\Lambda_i, \qquad i.e. \quad r_\alpha=\Lambda_{2^\alpha-2}.$$

Proof.—All possible trees of altitude $\alpha+1$ are subordinates of the plenary tree $2^{\alpha+1}$. Moreover, Theorem 7 implies that if we operate with Λ on a term of a ψ-polynomial ψ_P and the resulting polynomial ϕ differs from ψ_P then ϕ is the ψ-polynomial of a first subordinate of P. Thus all index ψ-polynomials of degree $\alpha+1$ can be obtained by operating with Λ on $2^{\alpha+1}\ \lambda^{\alpha+1}$, the index ψ-polynomial of the plenary tree $2^{\alpha+1}$, and on terms of the derived polynomials in such a way as to leave in each resulting polynomial a term in $\lambda^{\alpha+1}$. We can obtain all these polynomials in the following way: first operate with Λ on the leading terms only and obtain the sequence of ψ-polynomials of the first, second, . . . , $(2^\alpha-1)th$ principal subordinates of $2^{\alpha+1}$:

$$2^{\alpha+1}\lambda^{\alpha+1}, \quad (2^{\alpha+1}-2)\lambda^{\alpha+1}+\lambda^\alpha, \quad (2^{\alpha+1}-4)\lambda^{\alpha+1}+2\lambda^\alpha, \quad \ldots,$$
$$(2^{\alpha+1}-2i)\lambda^{\alpha+1}+i\lambda^\alpha, \quad \ldots, \quad 4\lambda^{\alpha+1}+(2^\alpha-2)\lambda^\alpha, \quad 2\lambda^{\alpha+1}+(2^\alpha-1)\lambda^\alpha.$$

Now, from each $(2^{\alpha+1}-2i)\lambda^{\alpha+1}+i\lambda^\alpha$ we can obtain all ψ-polynomials of degree $\alpha+1$ with leading term $(2^{\alpha+1}-2i)\lambda^{\alpha+1}$ by leaving the term in $\lambda^{\alpha+1}$ alone and operating with Λ on $i\lambda^\alpha$ and on other resulting terms. But, by the definition of Λ-value, we can obtain in this manner exactly Λ_i polynomials. Hence $r_{\alpha+1}=\sum_{i=0}^{2^\alpha-1}\Lambda_i$. Now, by the Lemma $\sum_{i=0}^{2^{\alpha-1}-1}\Lambda_i=\Lambda_{2^\alpha-2}$ and so $r_\alpha=\Lambda_{2^\alpha-2}$.

For $\alpha=0,\ 1,\ 2,\ 3,\ 4,\ 5,\ 6,\ 7,\ \ldots$
$r_\alpha=1,\ 1,\ 2,\ 6,\ 26,\ 166,\ 1626,\ 25510,\ \ldots$.

Let

$$f(x)=\Lambda_0+\Lambda_1x+\Lambda_2x^2+\ldots.$$

Then

$$\frac{f(x)}{1-x}=(\Lambda_0+\Lambda_1x+\Lambda_2x^2+\ldots)(1+x+x^2+\ldots)$$
$$=\Lambda_0+(\Lambda_0+\Lambda_1)x+(\Lambda_0+\Lambda_1+\Lambda_2)x^2+\ldots$$
$$=\Lambda_0+\Lambda_2x+\Lambda_4x^2+\ldots=\Lambda_1+\Lambda_3x+\Lambda_5x^2+\ldots.$$

Hence

$$f(x)=\frac{f(x^2)}{1-x^2}+\frac{xf(x^2)}{1-x^2},$$

i.e.

$$f(x)=\frac{f(x^2)}{1-x}.$$

This functional equation is easily solved by iteration:

$$f(x)=\frac{1}{1-x}f(x^2)=\frac{1}{1-x}\cdot\frac{1}{1-x^2}f(x^4)$$

$$=\frac{1}{1-x}\cdot\frac{1}{1-x^2}\cdot\frac{1}{1-x^4}f(x^8)=\ldots=\prod_{i=0}^{\infty}(1-x^{2^i})^{-1}$$

$$=(1+x+x^2+x^3+x^4+\ldots)(1+x^2+x^4+\ldots)(1+x^4+x^8+\ldots)$$
$$(1+x^8+\ldots)\ldots.$$

Thus r_a $(a>0)$ is the coefficient of x^{2a-2} in the Maclaurin expansion of this function. Alternatively, r_a is the coefficient of x^{2a} in $1+x^2f(x)$ (all a).

(The preceding paragraph on the generating function $f(x)$ was communicated to me by Dr Etherington.)

Finally we prove a formula for π_P, the number of trees (or indices) concordant to a given tree P. This formula is derived from a similar formula (for the number of indices having the same given palindromic index ψ-polynomial) communicated to me as a conjecture by Dr Etherington.

THEOREM 10.—If P is a given tree and

$$\psi_P=\sum_i \tau_i\lambda^i, \qquad \theta_P=\sum_i \pi_i\lambda^i$$

then

$$\pi_P=\prod_i\binom{\tau_i+\pi_i}{\pi_i}.$$

Proof.—Use induction on the altitude of P. The formula obviously holds for altitude 1. Assume that it holds for trees of altitudes less than a. If P is of altitude a then $\psi_Q=\psi_P-(2\lambda-1)\frac{1}{2}\tau_a\lambda^{a-1}$ is the ψ-polynomial of the $(\frac{1}{2}\tau_a)th$ principal subordinate of P, a tree of altitude $a-1$. Every tree whose ψ-polynomial is equal to ψ_Q has $\tau_{a-1}+\frac{1}{2}\tau_a$ free ends at its maximal altitude $a-1$. To obtain all trees concordant to ψ_P we join the nodes of $\frac{1}{2}\tau_a$ forks in all possible manners to these free ends. This can be done in $\binom{\tau_{a-1}+\frac{1}{2}\tau_a}{\frac{1}{2}\tau_a}$ distinct ways for each tree whose ψ-polynomial is equal to ψ_Q.

Hence $\pi_P = \binom{\tau_{a-1} + \frac{1}{2}\tau_a}{\frac{1}{2}\tau_a} . \pi_Q$. But, since $\psi_P = (2\lambda - 1)\theta_P + 1$, $\frac{1}{2}\tau_a = \pi_{a-1}$ and therefore $\pi_P = \binom{\tau_{a-1} + \pi_{a-1}}{\pi_{a-1}} . \pi_Q$. Now, by the induction hypothesis $\pi_Q = \prod_{i=1}^{a-2} \binom{\tau_i + \pi_i}{\pi_i}$. The result follows.

Example.—To find the number of trees concordant to 3.3.4.

$$\psi_{3.3.4} = 8\lambda^7 + 12\lambda^6 + 10\lambda^5 + 5\lambda^4 + \lambda^3,$$
$$\theta_{3.3.4} = 4\lambda^6 + 8\lambda^5 + 9\lambda^4 + 7\lambda^3 + 4\lambda^2 + 2\lambda + 1.$$

λ^i :	λ^6,	λ^5,	λ^4,	λ^3;	For all other terms either τ_i or π_i is 0 and thus $\binom{\tau_i + \pi_i}{\pi_i} = 1$.
τ_i :	12,	10,	5,	1;	
π_i :	4,	8,	9,	7.	

$$\pi_{3.3.4} = \binom{16}{4}\binom{18}{8}\binom{14}{9}\binom{8}{7} = 1\ 275\ 507\ 192\ 960.$$

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XII.—Solution of the Equation $ze^z = a$.*

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Synopsis

The roots of the equation $ze^z = a$ are of importance in several theories. Various authors have studied certain of their properties over more than a century. Here we solve the equation, in the sense that we define the sequence $\{Z_n\}$ of roots and, except for a small, finite number of values of n, find a rapidly convergent series for Z_n. The terms in this series are alternately real and purely imaginary and so the series is very convenient for calculation. For the few remaining roots, we give practicable methods of numerical calculation and supply an auxiliary table.

The main results of this article have been announced without proof or details in Wright 1959.

Introduction

1. The equation

$$ze^z = a \qquad (a \neq 0), \tag{1.1}$$

or some trivial transform of it, plays a part in the iteration of the exponential function (Euler 1927, Eisenstein 1844, Wright 1947) and in the theory and various applications of certain difference-differential equations (Polossuchin 1910, Schürer 1912, 1913, Bellman 1949, Wright 1955). For this reason, several authors (Hayes 1950, 1952, Lémeray 1896, 1897, Polossuchin 1910, Wright 1955) have studied properties of its roots. In particular, the least upper bound of the real parts of the roots is of importance in stability theory and has received special attention (Hayes 1950, Wright 1955).

The equation (1.1) is equivalent to

$$z + \log z = \log a, \tag{1.2}$$

where we consider $\log a$ many-valued and fix the value of $\log z$ by cutting the z-plane along the real axis from $-\infty$ to 0. In the interior of the cut-plane we suppose that

$$-\pi < \arg z = \mathcal{I}(\log z) < \pi.$$

We write $z = x + iy$ and $w = u + iv$, where

$$w = w(z) = z + \log z$$

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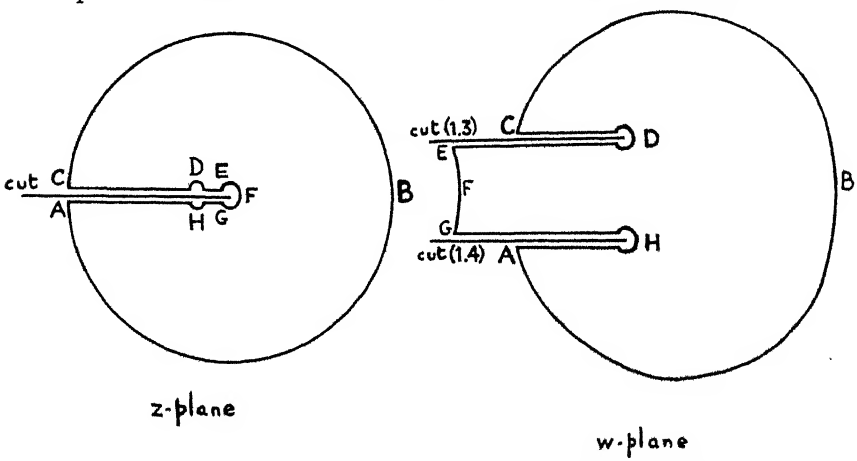
and x, y, u, v are real. We have

$$\frac{dw}{dz}=\frac{z+1}{z}$$

and this vanishes at $z=-1$. The upper edge of the cut in the z-plane corresponds to a cut in the w-plane along the semi-infinite straight line on which

$$u \leqslant -1, \qquad v=\pi, \tag{1.3}$$

the part on which $x < -1$ corresponding to the upper edge of the cut (1.3) and the part on which $-1 < x < 0$ to the lower edge. Similarly the lower



edge of the cut in the z-plane corresponds to both edges of a cut in the w-plane on which

$$u \leqslant -1, \qquad v=-\pi. \tag{1.4}$$

The points at which $w=-1\pm i\pi$ are branch points in the w-plane.

We take K, d two positive numbers, of which K is large and d small. $\mathscr{C}_z$ is the closed contour in the cut z-plane consisting of

(i) the circle $|z|=K$, described counter-clockwise,

(ii) the upper edge of the cut from $z=-K$ to $z=-d$ indented above $z=-1$ by a semicircle of radius d,

(iii) the circle $|z|=d$, described clockwise, and

(iv) the lower edge of the cut from $z=-d$ to $z=-K$ indented as in (ii), but below $z=-1$. The transform of $\mathscr{C}_z$ in the w-plane is $\mathscr{C}_w$, shown in the figure, in which corresponding points in the two planes have the same letters.

Now let w_0 be any fixed value of w not on either of the cuts (1.3) and (1.4). A little consideration shows that, in the w-plane, as $K\to\infty$, the

curve ABC recedes to infinity and, as $d \to 0$, the curve EFG recedes to infinity and the loops at D and H shrink inwards onto the ends of the cuts. Hence, by taking K large enough and d small enough, we can ensure that $\mathscr{C}_w$ encloses w_0 in its interior. Then, as z goes once counter-clockwise round $\mathscr{C}_z$, w goes once counter-clockwise round $\mathscr{C}_w$ and arg $(w - w_0)$ increases by 2π. Hence there is just one z_0 within $\mathscr{C}_z$ such that $w(z_0) = w_0$. Thus in the interior of the cut w-plane, the inverse function $z = z(w)$ is uniquely defined.

The function $z = z(w)$ can be continued analytically across either of the cuts in the w-plane onto another sheet of the corresponding Riemann surface. For example, if we cross the cut (1.3) downwards, we pass into a sheet in which the branch points are at $-1 + \pi i$ and $-1 + 3\pi i$. (If we encircle the first of these counter-clockwise and again cross the cut (1.3) downwards, we are back in the original sheet.) In every sheet of the Riemann surface the branch points lie on the straight line $u = -1$.

Now suppose that w_0 is any point in the interior of the original cut w-plane. Let us draw a circle with centre w_0 and circumference passing through the nearer of the two branch points $-1 \pm \pi i$, so that its radius is $\min |-1 \pm \pi i - w_0|$. This circle may have part of its interior in another sheet of the Riemann surface, but it cannot have any branch point in its interior. If w_1 is any point within the circle, *i.e.* if

$$|w_1 - w_0| < \min |-1 \pm \pi i - w_0|, \tag{1.5}$$

we have the convergent Taylor expansion

$$z(w_1) = z(w_0) + \sum_{k=1}^{\infty} Q_k (w_1 - w_0)^k, \tag{1.6}$$

where Q_k is the value of $\frac{1}{k!}\frac{d^k z}{dw^k}$ at $w = w_0$. The value of $z(w_1)$ given by this formula is the unique value valid in the cut w-plane, provided the straight line joining w_0 and w_1 does not *cross* one of the cuts.

If w_0 or w_1 or both lie on one or other of the cuts (but *not* at a branch point), (1.6) is still true, provided we take the appropriate values of $z(w_1)$ and $z(w_0)$. If, for example, w_0 and w_1 are both on the same cut, we must suppose them on the same edge of the cut and so take $z(w_0)$ and $z(w_1)$ both less than -1 or both between -1 and 0.

Enumeration of the Roots

2. If $a = Ae^{i\alpha}$, where $-\pi < \alpha \leqslant \pi$, $A > 0$, we have

$$\log a = \log A + i(\alpha + 2n\pi)$$

in (1.2), where n is any integer. Taking this as w in the discussion of the

last section, we see that

$$Z_n = z\{\log A + i(\alpha + 2n\pi)\} \tag{2.1}$$

is a root of (1.1) and that, apart from an exceptional case, all the roots of (1.1) are given by (2.1), when n runs through all integers, positive, zero or negative.

The exceptional case arises when $w = \log a$ lies on one or other of the two cuts (1.3) and (1.4) in the w-plane. In this case $\alpha = \pi$, $n = 0$ or -1 and $\log A \leqslant -1$, *i.e.* a is real and $-e^{-1} \leqslant a < 0$. There are then two real negative values of z corresponding to the value $\log A + i\pi$ of w and the same two values of z corresponding to the value $\log A - i\pi$. We take one of these values as Z_{-1} and the other as Z_0. We have $Z_{-1} \neq Z_0$, unless $a = -e^{-1}$, when $Z_{-1} = Z_0 = -1$. This is the double root of (1.1) when $a = -e^{-1}$.

Since the positive half of the real axis in the z-plane corresponds to the whole real axis in the w-plane, Z_0 is real when $\alpha = 0$, *i.e.* when a is real and positive.

Apart from these, there are no other real roots. For all the non-real roots, we put

$$Z_n = X_n + iY_n = R_n e^{i\theta_n} \qquad (0 < |\theta_n| < \pi) \tag{2.2}$$

and have

$$Y_n + \theta_n = \alpha + 2n\pi \tag{2.3}$$

by equating imaginary parts of (1.2). Clearly Y_n and θ_n have the same sign. Thus Y_0 lies between 0 and α,

$$(2n-1)\pi + \alpha < Y_n < 2n\pi + \alpha \qquad (n \geqslant 1)$$

and

$$2n\pi + \alpha < Y_n < (2n+1)\pi + \alpha \qquad (n \leqslant -1).$$

These preliminary results are not new. For real a, they are well known and essentially due to Lémeray (1896, 1897). For complex a, they are given by Hayes (1950) in a different notation. These authors use (real variable) methods different from mine.

Three Expansions for $z(w_1) - z(w_0)$

3. In (1.6), let us write $z_0 = z(w_0)$, $\gamma = w_1 - w_0$ and $t = -(1+z_0)^{-1}$. We have

$$\frac{dz}{dw} = \frac{z}{1+z}, \qquad \frac{d}{dw} = \frac{z}{1+z}\frac{d}{dz}$$

and so

$$Q_1 = 1 + t, \qquad Q_k = \frac{t^2 + t^3}{k}\frac{d}{dt}Q_{k-1} \qquad (k \geqslant 2). \tag{3.1}$$

It follows from (3.1) that

$$Q_k = \sum_{h=k}^{2k-1} b_{kh} t^h \qquad (k \geqslant 2), \tag{3.2}$$

where all the b_{kh} are positive numbers depending only on k and h. Now let us suppose that $|z_0| > 1$. We have

$$Q_k = \sum_{h=k}^{2k-1} \frac{(-1)^h b_{kh}}{(z_0+1)^h} = \sum_{m=k}^{\infty} \frac{(-1)^m a_{mk}}{{z_0}^m} \qquad (k \geqslant 2)$$

(say), where a_{mk} is a positive number depending only on m and k. Again

$$Q_1 = \frac{z_0}{z_0+1} = 1 + \sum_{m=1}^{\infty} \frac{(-1)^m}{{z_0}^m},$$

so that $a_{m1} = 1$. Now, by (3.1),

$$kQ_k = \frac{d}{dw_0} Q_{k-1} = \frac{z_0}{z_0+1} \frac{d}{dz_0} Q_{k-1}$$

and so

$$k(1+z_0) \sum_{m=k}^{\infty} (-1)^m a_{mk} {z_0}^{-m} = \sum_{m=k-1}^{\infty} (-1)^{m+1} m a_{m,\,k-1} {z_0}^{-m}.$$

From this we deduce that

$$k a_{kk} = (k-1) a_{k-1,\,k-1} \tag{3.3}$$

and that

$$k a_{m+1,\,k} = k a_{mk} + m a_{m,\,k-1} \qquad (m \geqslant k). \tag{3.4}$$

Combined with the fact that $a_{m1} = 1$, these enable us to calculate the a_{mk} in succession. We can also establish by induction on k and m that

$$a_{mk} \leqslant \frac{2^k (m+k-1)!}{(m-k)!\,(2k-1)!} \qquad (k \geqslant 1,\ m \geqslant k),$$

where, as usual, 0! denotes 1.

We have now by (1.6)

$$\begin{aligned} z(w_1) &= z_0 + \sum_{k=1}^{\infty} Q_k \gamma^k \\ &= z_0 + \gamma + \sum_{k=1}^{\infty} \gamma^k \sum_{m=k}^{\infty} (-1)^m a_{mk} {z_0}^{-m}. \end{aligned} \tag{3.5}$$

The double series is absolutely convergent provided that

$$\sum_{k=1}^{\infty} |2\gamma|^k \sum_{m=k}^{\infty} \frac{(m+k-1)!}{(m-k)!\,(2k-1)!} |z_0|^{-m} \tag{3.6}$$

is convergent. Since $|z_0| > 1$, we have

$$\frac{|z_0|^k}{(|z_0|-1)^{2k}} = \frac{|z_0|^{-k}}{(1-|z_0|^{-1})^{2k}} = \sum_{m=k}^{\infty} \frac{(m+k-1)!}{(m-k)!\,(2k-1)!} |z_0|^{-m}$$

and so the double series (3.6) is convergent, provided that both of

$$|z_0| > 1 \qquad |2\gamma z_0| < (|z_0|-1)^2 \tag{3.7}$$

are satisfied. When these are true, we can change the order of summation in (3.5) and find that

$$z(w_1) = z_0 + \gamma + \sum_{m=1}^{\infty} (-1)^m P_m(\gamma) z_0^{-m}, \tag{3.8}$$

where

$$P_m(\gamma) = \sum_{k=1}^{m} a_{mk} \gamma^k$$

is a polynomial in γ with positive coefficients. We have $P_1(\gamma) = \gamma$ and, by (3.3) and (3.4),

$$P_m(\gamma) = P_{m-1}(\gamma) + (m-1) \int_0^{\gamma} P_{m-1}(\sigma)\, d\sigma.$$

Hence, in particular,

$$P_2 = \gamma + \frac{1}{2}\gamma^2, \qquad P_3 = \gamma + \frac{3}{2}\gamma^2 + \frac{1}{3}\gamma^3, \qquad P_4 = \gamma + 3\gamma^2 + \frac{11}{6}\gamma^3 + \frac{1}{4}\gamma^4,$$

$$P_5 = \gamma + 5\gamma^2 + \frac{35}{6}\gamma^3 + \frac{25}{12}\gamma^4 + \frac{1}{5}\gamma^5.$$

We can deduce from (3.3) and (3.4) and the fact that $a_{m1} = 1$ that, for any ξ,

$$\xi \prod_{s=1}^{m-1} (1 + s\xi) = \sum_{k=1}^{m} k!\, a_{mk} \xi^k, \tag{3.9}$$

so that the left-hand side of (3.9) is a generating function of $k!\, a_{mk}$.

Again, provided that $2|t\gamma|(1+|t|) < 1$, we can deduce from (1.6) that

$$z(w_1) = z_0 + \sum_{h=0}^{\infty} t^h \chi_h(\gamma), \tag{3.10}$$

where

$$\chi_0 = \chi_1 = \gamma, \qquad \chi_h(\gamma) = \sum_{\frac{1}{2}h < k < h} b_{kh} \gamma^k \qquad (h > 2)$$

and

$$\chi_h'(\gamma) = (h-1)\chi_{h-1}(\gamma) + (h-2)\chi_{h-2}(\gamma) \qquad (h > 2).$$

But, as I have found no use for (3.9) or (3.10) in my present problem, I omit the proofs, which are in any case sufficiently obvious.

Calculation of the Distant Roots

4. For every positive integer n we write

$$H = 2n\pi + \alpha - \tfrac{1}{2}\pi, \qquad \beta = \log (A/H).$$

If we put $z_0 = iH$ and $z_1 = Z_n$, we have

$$\gamma = w(Z_n) - w(iH) = (2n\pi + \alpha)i + \log A - iH - \tfrac{1}{2}\pi i - \log H = \beta,$$

which is real. If we use these values in the expansion (3.8), we have

$$Z_n = iH + \beta + \sum_{m=1}^{\infty} i^m P_m(\beta) H^{-m}, \tag{4.1}$$

provided that (3.7) and (1.5) are satisfied, that is

$$2H \mid \beta \mid < (H-1)^2 \tag{4.2}$$

and

$$\mid \beta \mid < \mid w(iH) + 1 - \pi i \mid,$$

which is

$$(\log A)^2 < (H - \tfrac{1}{2}\pi)^2 + 2(1 + \log A) \log H + 1. \tag{4.3}$$

Both conditions are obviously satisfied for large enough n.

Since $Z_n = X_n + iY_n$, we have from (4.1)

$$Y_n = H + \eta, \tag{4.4}$$

where

$$\eta = \sum_{j=0}^{\infty} (-1)^j P_{2j+1}(\beta) H^{-2j-1} \tag{4.5}$$

and

$$X_n = \beta + \sum_{j=1}^{\infty} (-1)^j P_{2j}(\beta) H^{-2j}. \tag{4.6}$$

If we wish to calculate X_n but not Y_n, as is often the case in applications, we use (4.6). If, however, we wish to calculate both X_n and Y_n we can shorten our work by proceeding as follows. By (2.2) and (2.3), we have

$$X_n = Y_n \cot \theta_n = Y_n \cot (\alpha - Y_n) = (H + \eta) \tan \eta. \tag{4.7}$$

We now calculate η from (4.5) and have Y_n and X_n from (4.4) and (4.7). Since $\eta = O(\beta/H)$, tables of tan η may not enable us to calculate X_n from (4.7) with the accuracy we require for large H. If so, we use

$$X_n = (H + \eta) \tan \eta = \eta(H + \eta)(1 + \frac{\eta^2}{3} + \frac{2\eta^4}{15} + \frac{17\eta^6}{315} + O(\eta^8)). \tag{4.8}$$

For negative n, we write

$$H = -2n\pi - \tfrac{1}{2}\pi - \alpha > 0, \qquad \beta = \log (A/H), \qquad z_0 = -iH.$$

If (4.2) and (4.3) are satisfied, we have

$$Y_n = -H - \eta, \tag{4.9}$$

where η is defined in terms of H and β by (4.5). The formulæ (4.6), (4.7) and (4.8) are still true. Again, we may use (4.6) to calculate Y_n or (4.5), (4.9) and (4.8) to calculate both Y_n and X_n.

If we are to calculate η from a reasonable number of terms of (4.5), it is essential that β/H be fairly small. The conditions (4.2) and (4.4) are then certainly satisfied.

Two Expansions for $z(w)$ Valid in Particular Regions

5. For small values of $|a|$, it is well known (Hurwitz and Courant 1929) that one solution z of (1.1) has the expansion

$$\sum_{m=1}^{\infty} \frac{(-1)^{m-1} m^{m-1} a^m}{m!}.$$

This series converges for $|a| < e^{-1}$. It is easy to deduce that, when $u < -1$ and $-\pi \leqslant v \leqslant \pi$, *i.e.* when w lies between the two cuts in the w-plane, our inverse function $z(w)$ has the expansion

$$z(w) = \sum_{m=1}^{\infty} \frac{(-1)^{m-1} m^{m-1} e^{mw}}{m!}. \tag{5.1}$$

Near the branch point $w_0 = -1 + i\pi$, we write

$$z = -1 + \zeta, \qquad w = w_0 - \tfrac{1}{2}\omega^2,$$

so that

$$\zeta + \log(1 - \zeta) = -\tfrac{1}{2}\omega^2, \tag{5.2}$$

where the value of the logarithm is small when ζ is small. From (5.1), we have

$$\omega^2 = \zeta^2\left(1 + \frac{2}{3}\zeta + \frac{1}{2}\zeta^2 + \frac{2}{5}\zeta^3 + \ldots\right)$$

and we may invert this series and deduce that

$$z + 1 = \zeta = \sum_{m=1}^{\infty} c_m \omega^m, \tag{5.3}$$

where $c_1 = \pm 1$. Detailed consideration of the z, w mapping in the neighbourhood of w_0 shows us that we may take $c_1 = 1$, provided that we take that value of

$$\omega = \sqrt{(2w_0 - 2w)}$$

which has a positive imaginary part when w does not lie on the cut ending at w_0. When w lies on the top edge of the cut, ω is real and negative; when w lies on the bottom edge, ω is real and positive. Again, if we encircle w_0 twice counter-clockwise we pass from our cut plane into a sheet of the Riemann surface in which the branch points are at w_0 and

$w_0 + 2\pi i$, and then back into our original cut-plane. The singularities of $\zeta = \zeta(\omega)$ with smallest modulus therefore lie at

$$\omega = (2\pi)^{\frac{1}{2}}\,(1 \pm i)$$

and so the radius of convergence of (5.3) is $2\sqrt{\pi}$. Thus the expansion (5.3) is valid for $|w - w_0| < 2\pi$. It is readily verified that the same expansion of $z+1$ is valid with

$$\omega = \sqrt{(2\bar{w}_0 - 2w)},$$

the value for which $\mathscr{I}(\omega) < 0$ being chosen, provided that $|w - \bar{w}_0| < 2\pi$.

We have thus found expansions of z valid near the two branch points. It remains to calculate the leading c_m. Differentiating (5.2), we have

$$\zeta \frac{d\zeta}{d\omega} = \omega(1 - \zeta)$$

and so

$$\left(\sum_{h=1}^{\infty} c_h \omega^h\right)\left(\sum_{k=1}^{\infty} k c_k \omega^{k-1}\right) = \omega - \sum_{m=2}^{\infty} c_{m-1}\omega^m.$$

Equating coefficients of w^m, we find that

$$\sum_{k=1}^{m} k c_k c_{m-k+1} = -c_{m-1} \qquad (m > 2).$$

Since $c_1 = 1$, we find from this that $c_2 = -\frac{1}{3}$ and that

$$c_m = -\frac{c_{m-1}}{m+1} - \frac{1}{2}\sum_{k=2}^{m-1} c_k c_{m-k+1} \qquad (m > 3).$$

From this we can calculate $c_3, c_4, \ldots$ in succession and we have finally

$$z = -1 + \omega - \frac{\omega^2}{3} + \frac{\omega^3}{36} + \frac{\omega^4}{270} + \frac{\omega^5}{4320} - \frac{\omega^6}{17010} - \frac{139\omega^7}{5443200} - \frac{\omega^8}{204120} - \frac{571\omega^9}{2351462400} \ldots, \tag{5.4}$$

provided that $|\omega| < 2\sqrt{\pi}$.

The Near Roots

6. The method of § 4 suffices to calculate Z_n for all but a finite (usually quite small) number of values of n. For any particular a, it is easy to determine for just which n it is inapplicable (or only applicable with excessive labour). For these n, we have to calculate $z(w)$. where $w = \log A + (\alpha + 2n\pi)i$.

If w lies between the cuts in the w-plane, and (say) $u \leqslant -2$, we use

the series (5.1). If w lies near one of the branch points $-1 \pm \pi i$, we use (5.4). Otherwise we find a first approximation z_0 to $z(w)$ and improve it as follows.

We have

$$\frac{dw}{dz} = 1 + \frac{1}{z}, \qquad \frac{dz}{dw} = \frac{z}{1+z} = 1 - \frac{1}{1+z}$$

and so, to a first approximation,

$$\left.\begin{aligned} \delta x &= \{1 - \lambda(1+x)\}\delta u - y\lambda\delta v, \\ \delta y &= y\lambda\delta u + \{1 - \lambda(1+x)\}\delta v, \end{aligned}\right\} \tag{6.1}$$

where $\lambda^{-1} = (1+x)^2 + y^2$. If we have an approximation z_0 to Z_n we can calculate

$$\delta u + i\delta v = w(Z_n) - w(z_0) = \log A + \alpha i + 2n\pi i - w(z_0)$$

and so use (6.1) to determine a correction $\delta x + i\delta y$ to z_0. By (1.6) and (3.1), we have

$$\delta z = z_0(1+z_0)^{-1}\delta w + o\{(\delta w)^2 z_0(1+z_0)^{-3}\}.$$

Hence the repeated use of (6.1) leads to a rapidly convergent sequence of approximations, unless z_0 is near -1. In the latter case, w is near one of $-1 \pm \pi i$ and (5.4) applies.

If $|w| > 4$, but w does not lie between the cuts in the w-plane, we can take $z_0 = w - \log w$, where $\log w$ has its principal value. With this value of z_0, the next approximation z_1 will be correct to at least one decimal place and the subsequent approximations $z_2, z_3, \ldots$ converge rapidly.

When $|w| < 4$, but w is not near one of the branch points $-1 \pm \pi i$, we can use the table of values of u, v to find a suitable value for z_0. Since z_0 will not be near -1, the sequence of approximations converges fairly rapidly.

Alternatively we can use drawing to obtain our first approximation. Given u, v we have to solve

$$x + \log r = u, \qquad y + \theta = v,$$

where $r^2 = x^2 + y^2$, $\tan\theta = y/x$. To solve these graphically, we require (i) a sheet of "radial" graph paper (concentric circles and radii) called the (x, y)-plane and (ii) a sheet of tracing paper (the (X, Y)-plane) on which the lines $X = -\log k$ and $Y = -h$ are drawn, where k runs through the values of the radii of the circles and h through the values of θ corresponding to the radial lines. We place the origin of the (X, Y)-plane at the point (u, v) on the (x, y)-plane, make the axes parallel and then plot on a second sheet of tracing paper (the second (x, y)-plane) over the first the intersections of $X = -\log k$ with $r = k$ and those of $Y = -h$ with $\theta = h$.

X Y	−4·0	−3·5	−3·0	−2·5	−2·0	−1·5	−1·0	−·9	−·8	−·7	−·6	−·5	−·4	−·3	−·2	
·0	−2·614	−2·247	−1·901	−1·584	−1·307	−1·095	−1·000	−1·005	−1·023	−1·057	−1·111	−1·193	−1·316	−1·504	−1·809	−
	3·142	3·142	3·142	3·142	3·142	3·142	3·142	3·142	3·142	3·142	3·142	3·142	3·142	3·142	3·142	
·1	−2·613	−2·247	−1·901	−1·583	−1·306	−1·092	−0·995	−0·999	−1·015	−1·047	−1·097	−1·174	−1·286	−1·451	−1·698	−
	3·217	3·213	3·208	3·202	3·192	3·175	3·142	3·131	3·117	3·100	3·076	3·044	2·997	2·920	2·778	
·2	−2·612	−2·246	−1·899	−1·581	−1·302	−1·086	−0·980	−0·981	−0·993	−1·017	−1·058	−1·119	−1·205	−1·320	−1·463	−
	3·292	3·285	3·275	3·262	3·242	3·209	3·144	3·123	3·097	3·063	3·020	2·961	2·878	2·754	2·556	
·3	−2·611	−2·244	−1·896	−1·577	−1·296	−1·075	−0·957	−0·953	−0·957	−0·972	−0·999	−1·039	−1·093	−1·157	−1·220	−
	3·367	3·356	3·342	3·322	3·293	3·244	3·150	3·120	3·083	3·037	2·978	2·901	2·798	2·656	2·459	
·4	−2·609	−2·241	−1·893	−1·571	−1·287	−1·060	−0·926	−0·915	−0·912	−0·915	−0·927	−0·946	−0·970	−0·993	−1·005	−
	3·442	3·428	3·409	3·383	3·344	3·281	3·161	3·123	3·078	3·022	2·954	2·867	2·756	2·614	2·434	
·5	−2·606	−2·237	−1·888	−1·564	−1·277	−1·042	−0·888	−0·871	−0·858	−0·851	−0·847	−0·847	−0·846	−0·839	−0·819	−
	3·517	3·500	3·476	3·444	3·397	3·320	3·178	3·134	3·083	3·021	2·947	2·856	2·746	2·611	2·451	
·6	−2·603	−2·233	−1·882	−1·556	−1·264	−1·020	−0·846	−0·821	−0·800	−0·781	−0·764	−0·747	−0·727	−0·699	−0·658	−
	3·593	3·572	3·544	3·506	3·450	3·361	3·201	3·154	3·098	3·033	2·956	2·866	2·759	2·634	2·493	
·7	−2·599	−2·228	−1·875	−1·546	−1·249	−0·996	−0·801	−0·769	−0·739	−0·710	−0·681	−0·651	−0·615	−0·572	−0·517	−
	3·668	3·644	3·612	3·569	3·505	3·405	3·231	3·181	3·123	3·056	2·979	2·891	2·790	2·676	2·549	
·8	−2·594	−2·222	−1·867	−1·535	−1·233	−0·969	−0·753	−0·714	−0·677	−0·639	−0·600	−0·558	−0·512	−0·457	−0·393	−
	3·744	3·717	3·681	3·632	3·561	3·452	3·267	3·215	3·156	3·090	3·014	2·929	2·834	2·730	2·616	
·9	−2·589	−2·215	−1·858	−1·523	−1·215	−0·941	−0·703	−0·659	−0·614	−0·569	−0·521	−0·471	−0·415	−0·353	−0·281	−
	3·820	3·790	3·750	3·696	3·619	3·501	3·309	3·256	3·197	3·132	3·059	2·978	2·889	2·793	2·689	
1·0	−2·583	−2·208	−1·849	−1·509	−1·195	−0·911	−0·653	−0·603	−0·553	−0·501	−0·446	−0·388	−0·326	−0·257	−0·180	−
	3·897	3·863	3·820	3·761	3·678	3·554	3·356	3·304	3·246	3·182	3·111	3·034	2·951	2·862	2·768	
1·5	−2·548	−2·163	−1·790	−1·430	−1·084	−0·748	−0·411	−0·341	−0·269	−0·196	−0·120	−0·042	0·040	0·125	0·214	
	4·283	4·237	4·178	4·101	3·998	3·856	3·659	3·611	3·561	3·507	3·451	3·393	3·331	3·268	3·203	
2·0	−2·502	−2·106	−1·718	−1·336	−0·960	−0·584	−0·195	−0·115	−0·033	0·051	0·136	0·223	0·313	0·404	0·498	
	4·678	4·622	4·554	4·467	4·356	4·214	4·034	3·994	3·951	3·907	3·862	3·816	3·768	3·720	3·670	
2·5	−2·449	−2·041	−1·638	−1·237	−0·836	−0·430	−0·009	0·077	0·165	0·254	0·344	0·436	0·529	0·623	0·719	
	5·083	5·021	4·947	4·856	4·746	4·611	4·451	4·416	4·380	4·344	4·306	4·268	4·229	4·190	4·151	

Table of values of u, v, where $u+iv$

X / Y	−·1	·0	+·1	+·2	+·3	+·4	+·5	+·6	+·7	+·8	+·9	+1·0	+1·5	+2·0	+2·5	+3·0
·2·403		−2·203	−1·409	−0·904	−0·516	−0·193	0·089	0·343	0·577	0·795	1·000	1·905	2·693	3·416	4·099	·0
3·142		o	o	o	o	o	o	o	o	o	o	o	o	o	o	
-2·056	−2·303	−1·856	−1·298	−0·851	−0·486	−0·174	0·103	0·353	0·585	0·801	1·005	1·908	2·694	3·417	4·099	·1
2·456	1·671	0·885	0·564	0·422	0·345	0·297	0·265	0·242	0·224	0·211	0·200	0·167	0·150	0·140	0·133	
1·598	−1·609	−1·398	−1·063	−0·720	−0·405	−0·119	0·142	0·383	0·607	0·819	1·020	1·914	2·698	3·419	4·101	·2
2·234	1·771	1·307	0·985	0·788	0·664	0·581	0·522	0·478	0·445	0·419	0·397	0·333	0·300	0·280	0·267	
·1·251	−1·204	−1·051	−0·820	−0·557	−0·293	−0·039	0·201	0·428	0·643	0·847	1·043	1·925	2·704	3·423	4·104	·3
2·193	1·871	1·549	1·283	1·085	0·944	0·840	0·764	0·705	0·659	0·622	0·591	0·497	0·449	0·419	0·400	
·0·986	−0·916	−0·786	−0·605	−0·393	−0·170	0·054	0·273	0·485	0·688	0·885	1·074	1·940	2·713	3·429	4·107	·4
2·216	1·971	1·726	1·507	1·327	1·185	1·075	0·988	0·919	0·864	0·818	0·781	0·661	0·597	0·559	0·533	
·0·774	−0·693	−0·574	−0·419	−0·239	−0·046	0·153	0·353	0·549	0·742	0·929	1·112	1·958	2·723	3·436	4·112	·5
2·268	2·071	1·873	1·690	1·530	1·396	1·285	1·195	1·120	1·059	1·007	0·964	0·822	0·745	0·697	0·665	
·0·597	−0·511	−0·397	−0·258	−0·099	0·073	0·253	0·436	0·619	0·800	0·979	1·154	1·980	2·736	3·444	4·118	·6
2·336	2·171	2·006	1·849	1·707	1·583	1·476	1·385	1·309	1·244	1·188	1·140	0·981	0·891	0·836	0·797	
·0·447	−0·357	−0·247	−0·117	0·028	0·185	0·349	0·519	0·690	0·861	1·031	1·199	2·004	2·751	3·454	4·125	·7
2·413	2·271	2·129	1·992	1·866	1·752	1·651	1·562	1·485	1·419	1·361	1·311	1·137	1·037	0·973	0·929	
·0·315	−0·223	−0·115	0·007	0·143	0·288	0·442	0·600	0·761	0·923	1·086	1·247	2·031	2·767	3·465	4·133	·8
2·495	2·371	2·246	2·126	2·012	1·907	1·812	1·727	1·652	1·585	1·527	1·475	1·290	1·181	1·110	1·061	
·0·199	−0·105	0·001	0·119	0·247	0·385	0·529	0·679	0·831	0·986	1·141	1·297	2·059	2·785	3·477	4·142	·9
2·581	2·471	2·360	2·252	2·149	2·053	1·964	1·883	1·810	1·744	1·685	1·633	1·440	1·323	1·246	1·191	
·0·095	−0·000	0·105	0·220	0·343	0·474	0·612	0·754	0·899	1·047	1·197	1·347	2·089	2·805	3·491	4·151	1·0
2·670	2·571	2·471	2·373	2·279	2·190	2·107	2·030	1·960	1·896	1·838	1·785	1·588	1·464	1·381	1·322	
0·308	0·405	0·508	0·614	0·725	0·840	0·958	1·080	1·204	1·331	1·459	1·589	2·252	2·916	3·570	4·210	1·5
3·137	3·071	3·004	2·938	2·873	2·810	2·749	2·690	2·634	2·581	2·530	2·483	2·285	2·144	2·040	1·964	
0·594	0·693	0·794	0·898	1·004	1·113	1·223	1·336	1·451	1·567	1·685	1·805	2·416	3·040	3·664	4·282	2·0
3·621	3·571	3·521	3·471	3·422	3·373	3·326	3·279	3·234	3·190	3·148	3·107	2·927	2·785	2·675	2·588	
0·817	0·916	1·017	1·119	1·223	1·329	1·436	1·544	1·654	1·765	1·877	1·991	2·570	3·164	3·763	4·362	2·5
4·111	4·071	4·031	3·991	3·951	3·912	3·873	3·835	3·798	3·761	3·725	3·690	3·530	3·396	3·285	3·195	

$= x + iy + \log(x + iy)$.

These two sets of points lie on two curves whose intersection is the required solution in the (x, y)-plane. The first piece of tracing paper can be replaced by suitable markings on the edges of the drawing board and the use of a tee-square.

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